Integral Definida

Ejercicios resueltos

1 Resuelve las siguientes integrales definidas:

a) $\int_0^2 3x^2 dx$

b) $\int_0^1 e^x dx$

c) $\int_1^e \frac{1}{x} dx$

d) $\int_{-1}^1 \left(x+2x^2-x^3+5x^4\right) dx$

e) $\int_2^3 \frac{1}{\sqrt{x-1}} dx$

f) $\int_1^2 \frac{2x+1}{x^2+x} dx$

g) $\int_0^{2\pi} senx dx$

h) $\int_0^1 \frac{1}{1+x^2} dx$

i) $\int_2^5 \frac{1}{(x-1)\cdot(x+2)} dx$

j) $\int_0^1 \frac{x}{1+x^4} dx$

Solución

a) $\int_0^2 3x^2 dx = \left[x^3\right]_0^2 = \left(2^3\right)-\left(0^3\right) = 8-0 = 8$

b) $\int_0^1 e^x dx = \left[e^x\right]_0^1 = \left(e^1\right)-\left(e^0\right) = e-1$

c) $\int_1^e \frac{1}{x} dx = \left[\ln|x|\right]_1^e = \left(\ln|e|\right)-\left(\ln|1|\right) = 1-0 = 1$

d) $\int_{-1}^1 \left(x+2x^2-x^3+5x^4\right) dx = \left[\frac{1}{2}x^2+\frac{2}{3}x^3-\frac{1}{4}x^4+x^5\right]_{-1}^1 =$

$$= \left(\frac{1}{2}+\frac{2}{3}-\frac{1}{4}+1\right)-\left(\frac{1}{2}-\frac{2}{3}-\frac{1}{4}-1\right) = \frac{2}{3}+1+\frac{2}{3}+1 = \frac{4}{3}+2 = \frac{10}{3}$$

e) $\int_2^3 \frac{1}{\sqrt{x-1}} dx$

$$\int \frac{1}{\sqrt{x-1}}dx = \begin{pmatrix} t = x-1 \\ dt = dx \end{pmatrix} = \int \frac{1}{\sqrt{t}}dt = \int t^{-1/2}dt = \frac{t^{1/2}}{1/2} + C = 2\sqrt{x-1} + C$$

$$\int_2^3 \frac{1}{\sqrt{x-1}}dx = \left[2\sqrt{x-1}\right]_2^3 = \left(2\sqrt{2}\right) - \left(2\sqrt{1}\right) = 2\sqrt{2} - 2$$

f) $\int_1^2 \frac{2x+1}{x^2+x}dx$

$$\int \frac{2x+1}{x^2+x}dx = \int \frac{2x+1}{x\cdot(x+1)}dx = \int \frac{A}{x}dx + \int \frac{B}{x+1}dx$$

$$\frac{2x+1}{x\cdot(x+1)} = \frac{A}{x} + \frac{B}{x+1} \Rightarrow 2x+1 = A\cdot(x+1) + B\cdot x$$

$$\left.\begin{matrix} x = 0 \Rightarrow & 1 & = & A \\ x = -1 \Rightarrow & -1 & = & -B \end{matrix}\right\} \Rightarrow A = B = 1$$

$$\int_1^2 \frac{2x+1}{x^2+x}dx = \int_1^2 \frac{1}{x}dx + \int_1^2 \frac{1}{x+1}dx = \left[\ln|x| + \ln|x+1|\right]_1^2 =$$

$$= \left(\ln|2| + \ln|3|\right) - \left(\ln|1| + \ln|2|\right) = \ln 2 + \ln 3 - \ln 2 = \ln 3$$

g) $\int_0^{2\pi} senx dx = \left[-\cos x\right]_0^{2\pi} = \left(-\cos 2\pi\right) - \left(-\cos 0\right) = -1 + 1 = 0$

h) $\int_0^1 \frac{1}{1+x^2}dx = \left[arctagx\right]_0^1 = \left(arctag1\right) - \left(arctag0\right) = \frac{\pi}{4} - 0 = \frac{\pi}{4}$

i) $\int_2^5 \frac{1}{(x-1)\cdot(x+2)}dx$

$$\int \frac{1}{(x-1)\cdot(x+2)}dx = \int \frac{A}{x-1}dx + \int \frac{B}{x+2}dx$$

$$\frac{1}{(x-1)\cdot(x+2)} = \frac{A}{x-1} + \frac{B}{x+2} \Rightarrow 1 = A\cdot(x+2) + B\cdot(x-1)$$

$$\left.\begin{matrix} x = 1 \Rightarrow & 1 & = & 3A \\ x = -2 \Rightarrow & 1 & = & -3B \end{matrix}\right\} \Rightarrow A = \frac{1}{3} \qquad B = -\frac{1}{3}$$

$$\int_2^5 \frac{1}{(x-1)\cdot(x+2)}dx = \frac{1}{3}\int_2^5 \left(\frac{1}{x-1} - \frac{1}{x+2}\right)dx = \frac{1}{3}\left[\ln|x-1| - \ln|x+2|\right]_2^5 =$$

$$=\frac{1}{3}\left(\ln|4|-\ln|7|\right)-\frac{1}{3}\left(\ln|1|-\ln|4|\right)=\frac{1}{3}\ln 4-\frac{1}{3}\ln 7+\frac{1}{3}\ln 4=\frac{2}{3}\ln 4-\frac{1}{3}\ln 7$$

j) $\int_0^1 \frac{x}{1+x^4}dx$

$$\int \frac{x}{1+x^4}dx=\begin{pmatrix} t=x^2 \\ dt=2xdx \end{pmatrix}=\frac{1}{2}\int \frac{1}{1+t^2}dt=\frac{1}{2}arctg(t)+C=\frac{1}{2}arctg\left(x^2\right)+C$$

$$\int_0^1 \frac{x}{1+x^4}dx=\frac{1}{2}\left[arctg\left(x^2\right)\right]_0^1=\frac{1}{2}\left(arctg(1)\right)-\frac{1}{2}\left(arctg(0)\right)=\frac{1}{2}\cdot\frac{\pi}{4}-\frac{1}{2}\cdot 0=\frac{\pi}{8}$$

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2 Calcula el área de la región limitada por las siguientes gráficas:

a) $\left.\begin{array}{l} y=x+1 \\ y=0\ (EJE\ OX) \\ x=0 \\ x=1 \end{array}\right\}$

b) $\left.\begin{array}{l} y=x^2+1 \\ y=0\ (EJE\ OX) \\ x=1 \\ x=2 \end{array}\right\}$

c) $\left.\begin{array}{l} y=x^3 \\ y=0\ (EJE\ OX) \\ x=0 \\ x=2 \end{array}\right\}$

d) $\left.\begin{array}{l} y=x^2 \\ y=-x+2 \\ y=0\ (EJE\ OX) \end{array}\right\}$

e) $\left.\begin{array}{l} y=x^2-x-2 \\ y=0\ (EJE\ OX) \\ x=0 \\ x=1 \end{array}\right\}$

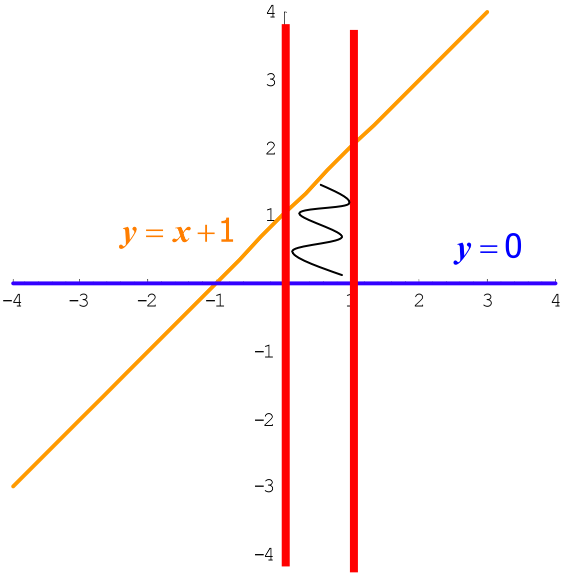
f) $\left.\begin{array}{l} y=\cos x \\ y=0\ (EJE\ OX) \\ x=\pi/2 \\ x=3\pi/2 \end{array}\right\}$

g) $\left.\begin{array}{l} y=x^2 \\ y=x \end{array}\right\}$

h) $\left.\begin{array}{l} y=-x^2+6x \\ y=x^2-2x \end{array}\right\}$

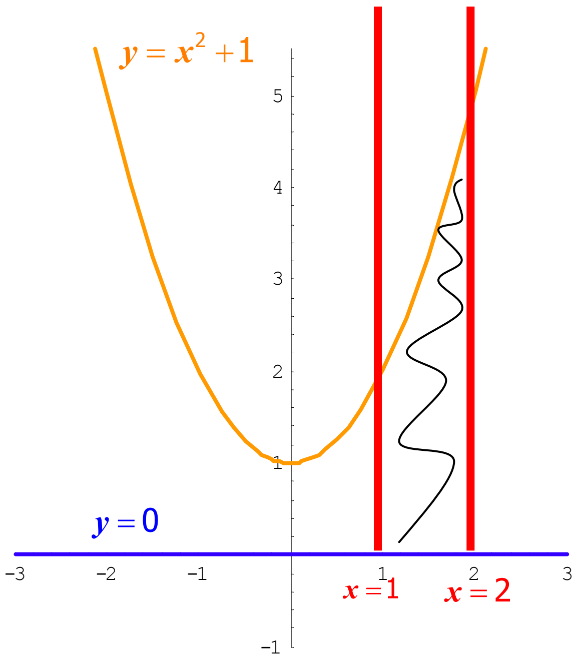
Solución

$$a)\quad \left.\begin{aligned} y &= x+1 \\ y &= 0\ (EJE\ \ OX) \\ x &= 0 \\ x &= 1 \end{aligned}\right\} \qquad A=\int_0^1 (x+1)\,dx=\left[\frac{1}{2}x^2+x\right]_0^1=\left(\frac{1}{2}+1\right)-(0)=\frac{3}{2}$$



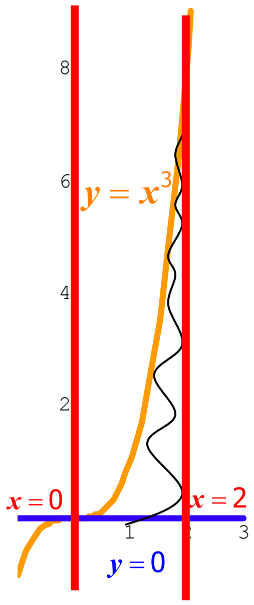
$$b)\quad \left.\begin{aligned} y &= x^2+1 \\ y &= 0\ (EJE\ \ OX) \\ x &= 1 \\ x &= 2 \end{aligned}\right\}$$

$$A=\int_1^2 (x^2+1)\,dx=\left[\frac{1}{3}x^3+x\right]_1^2=\left(\frac{8}{3}+2\right)-\left(\frac{1}{3}+1\right)=\frac{7}{3}+1=\frac{10}{3}$$

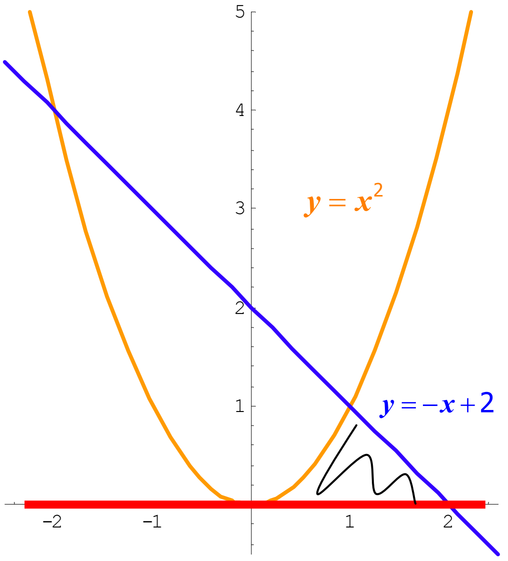


c) $\left.\begin{array}{l} y = x^3 \\ y = 0 \ (EJE\ OX) \\ x = 0 \\ x = 2 \end{array}\right\}$

$$A = \int_0^2 x^3 dx = \left[\frac{1}{4}x^4\right]_0^2 = \left(\frac{16}{4}\right) - 0 = 4$$



d) $\left.\begin{array}{l} y = x^2 \\ y = -x + 2 \\ y = 0 \ (EJE\ OX) \end{array}\right\}$



Puntos de corte:

$$x^2 = -x + 2 \Rightarrow x^2 + x - 2 = 0$$

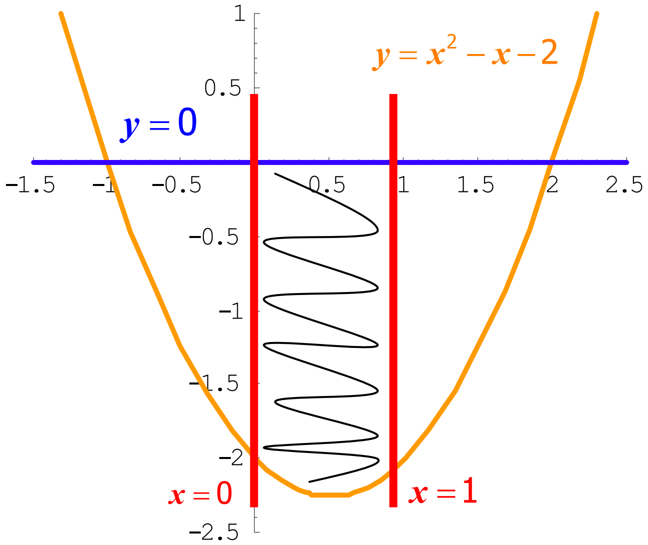
$$x = 1, -2$$

$$A = \int_0^1 x^2 dx + \int_1^2 (-x + 2)\, dx =$$

$$= \left[\frac{x^3}{3}\right]_0^1 + \left[-\frac{x^2}{2} + 2x\right]_1^2$$

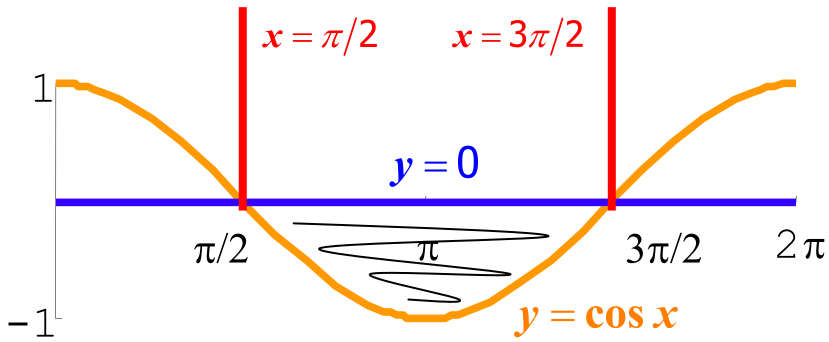
$$A = \left[\frac{1}{3} - 0\right] + \left[(-2 + 4) - \left(-\frac{1}{2} + 2\right)\right] = \frac{1}{3} + 2 - \frac{3}{2} = \frac{5}{6}$$

$$e)\quad \left.\begin{array}{l} y = x^2 - x - 2 \\ y = 0\ (EJE\ OX) \\ x = 0 \\ x = 1 \end{array}\right\}$$



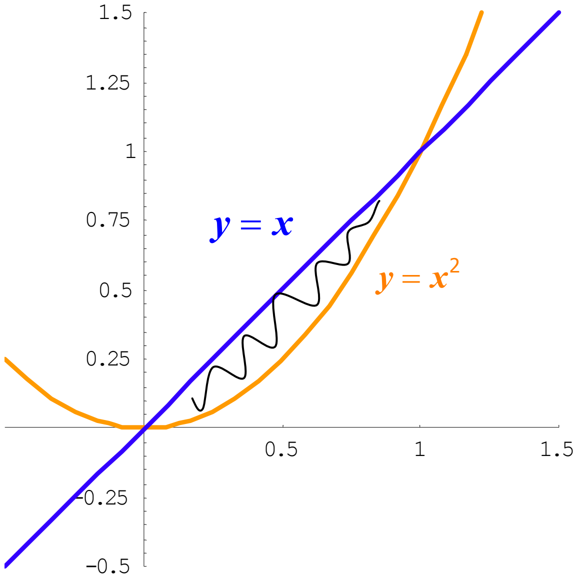
$$A = -\int_0^1 \left(x^2 - x - 2\right)dx = -\left[\frac{x^3}{3} - \frac{x^2}{2} - 2x\right]_0^1 = -\left[\left(\frac{1}{3} - \frac{1}{2} - 2\right) - 0\right] = \frac{13}{6}$$

$$f)\quad \left.\begin{array}{l} y = \cos x \\ y = 0\ (EJE\ OX) \\ x = \pi/2 \\ x = 3\pi/2 \end{array}\right\}$$



$$A = -\int_{\pi/2}^{3\pi/2} \cos x dx = -\left[senx\right]_{\pi/2}^{3\pi/2} = -\left[sen\left(\frac{3\pi}{2}\right) - sen\left(\frac{\pi}{2}\right)\right] = -\left[-1-1\right] = 2$$

g) $\left.\begin{array}{l} y = x^2 \\ y = x \end{array}\right\}$

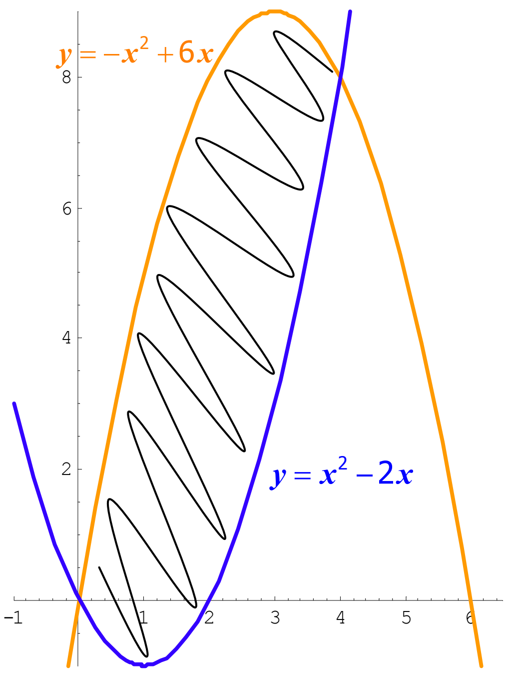


Puntos de corte:

$$x^2 = x \Rightarrow x^2 - x = 0 \Rightarrow x \cdot (x-1) = 0 \Rightarrow x = 0,1$$

$$A = \int_0^1 (x - x^2)\,dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \left[\left(\frac{1}{2} - \frac{1}{3}\right) - 0\right] = \frac{1}{6}$$

h) $\left.\begin{array}{l} y = -x^2 + 6x \\ y = x^2 - 2x \end{array}\right\}$



Puntos de corte:

$$-x^2 + 6x = x^2 - 2x \Rightarrow$$

$$\Rightarrow 2x^2 - 8x = 0 \Rightarrow x \cdot (x-4) = 0$$

$$x = 0,4$$

$$A = \int_0^4 \left[(-x^2 + 6x) - (x^2 - 2x)\right] dx =$$

$$= \int_0^4 (-2x^2 + 8x)\,dx =$$

$$= \left[-2\frac{x^3}{3} + 8\frac{x^2}{2}\right]_0^4 =$$

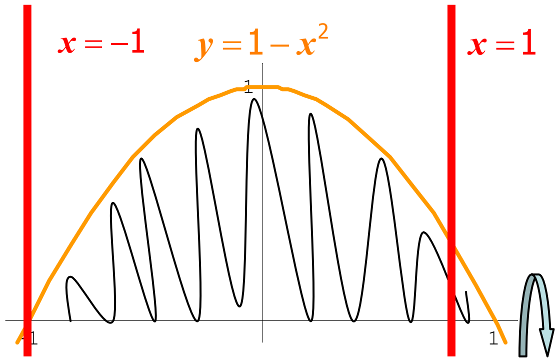
$$= \left[\left(-\frac{128}{3} + 64\right) - 0\right] = \frac{64}{3}$$

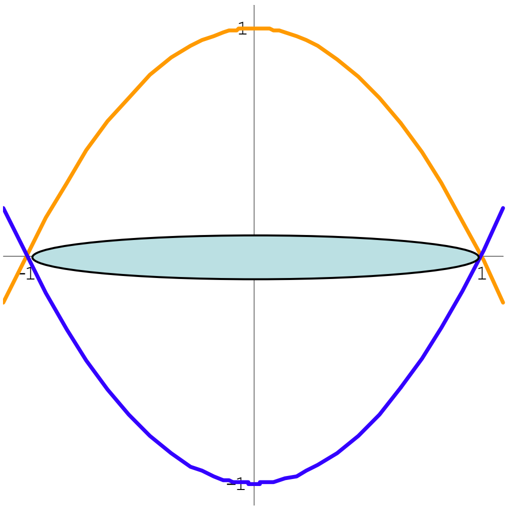
3 Calcula el volumen del sólido de revolución generado al girar alrededor del eje OX las siguientes gráficas:

$$a)\ \left.\begin{array}{l} y = 1 - x^2 \\ x = -1 \\ x = 1 \end{array}\right\} \qquad b)\ \left.\begin{array}{l} y = -\frac{1}{2}x^2 + 2x \\ y = \frac{1}{2}x \end{array}\right\} \qquad c)\ \left.\begin{array}{l} y = 3 \\ y = 0 \\ x = 0 \\ x = 5 \end{array}\right\}$$

Solución

$$a)\ \left.\begin{array}{l} y = 1 - x^2 \\ x = -1 \\ x = 1 \end{array}\right\}$$





$$V = \pi \int_{-1}^{1} \left(1 - x^2\right)^2 dx =$$

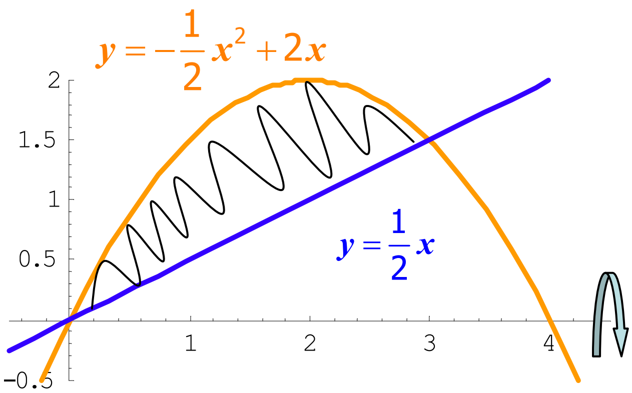
$$= \pi \int_{-1}^{1} \left(1 - 2x^2 + x^4\right) dx =$$

$$= \pi \left[x - \frac{2x^3}{3} + \frac{x^5}{5} \right]_{-1}^{1} =$$

$$= \pi \left(1 - \frac{2}{3} + \frac{1}{5} \right) - \pi \left(-1 + \frac{2}{3} - \frac{1}{5} \right) =$$

$$= \pi \left(2 - \frac{4}{3} + \frac{2}{5} \right) = \frac{16}{15}\pi$$

$$b)\ \left.\begin{array}{l} y = -\frac{1}{2}x^2 + 2x \\ y = \frac{1}{2}x \end{array}\right\}$$



Puntos de corte:

$$-\frac{x^2}{2}+2x=\frac{x}{2}\Rightarrow -x^2+4x=x\Rightarrow x^2-3x=0\Rightarrow x\cdot(x-3)=0\Rightarrow x=0,3$$

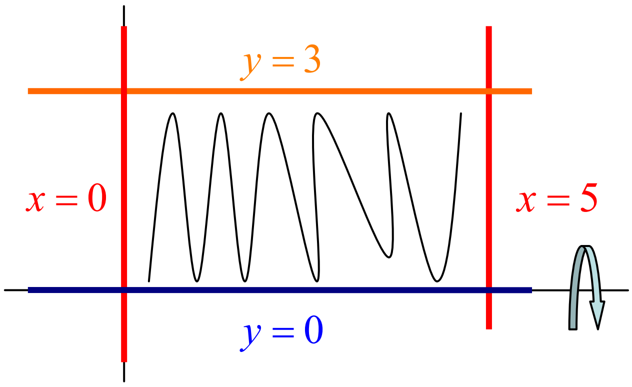
Se calcula el volumen generado por la gráfica de arriba (volumen lleno) y se le resta el volumen generado por la gráfica de abajo (volumen del agujero):

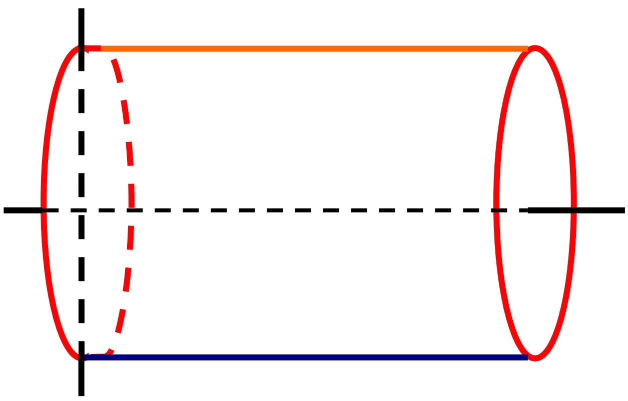
$$V=\pi\int_0^3\left(-\frac{x^2}{2}+2x\right)^2dx-\pi\int_0^3\left(\frac{x}{2}\right)^2dx=\pi\int_0^3\left(\frac{x^4}{4}-2x^3+4x^2\right)dx-\pi\int_0^3\frac{x^2}{4}dx=$$

$$=\pi\int_0^3\left(\frac{x^4}{4}-2x^3+\frac{15}{4}x^2\right)dx=\pi\left[\frac{x^5}{20}-\frac{x^4}{2}+\frac{5x^3}{4}\right]_0^3=$$

$$=\pi\left(\frac{243}{20}-\frac{81}{2}+\frac{135}{4}\right)-\pi(0)=\frac{243-810+675}{20}\pi=\frac{108}{20}\pi=\frac{27}{5}\pi$$

c) $\left.\begin{array}{l} y=3\\ y=0\\ x=0\\ x=5\end{array}\right\}$





$$V=\pi\int_0^5(3)^2dx=\pi\int_0^5 9dx=\pi[9x]_0^5=\pi(45-0)=45\pi$$