1 Resuelve la integral:

$$\int \frac{dx}{x^2 - 4}$$

SOLUCIÓN

$$\int \frac{dx}{x^2 - 4} = \int \frac{dx}{(x+2)(x-2)}$$

Utilizaremos el método de descomposición en fracciones simples:

$$\frac{1}{(x+2)(x-2)} = \frac{A}{x+2} + \frac{B}{x-2} = \frac{A(x-2) + B(x+2)}{(x+2)(x-2)}$$

Igualando los numeradores: $1 = A(x-2) + B(x+2)$, y dando a *x* los valores de las raíces reales del denominador, se obtienen valores para *A* y *B*:

$$x = 2 \Rightarrow B = \frac{1}{4} \quad , \quad x = -2 \Rightarrow A = -\frac{1}{4}$$

Luego, aplicando propiedades elementales de integración:

$$\int \frac{dx}{x^2 - 4} = \int \frac{-1/4}{x+2} dx + \int \frac{1/4}{x-2} dx = -\frac{1}{4} Log|x+2| + \frac{1}{4} Log|x-2| + C$$

2 Resuelve la integral:

$$\int \frac{Ln\,x}{\sqrt{1-x}}\,dx$$

SOLUCIÓN

Llamemos $I=\int \frac{Ln\,x}{\sqrt{1-x}}\,dx$

Aplicamos partes: $\left\{\begin{array}{l} Ln\,x=u \Rightarrow \frac{dx}{x}=du \\ \frac{dx}{\sqrt{1-x}}=dv \Rightarrow v=-2\sqrt{1-x} \end{array}\right\} \Rightarrow$

$$I=-2\sqrt{1-x}\;Ln\,x+2\int \frac{\sqrt{1-x}}{x}\,dx\,.$$

$$2\int \frac{\sqrt{1-x}}{x}\,dx=\left\{\begin{array}{l} 1-x=t^2 \\ -dx=2tdt \end{array}\right\}=-4\int \frac{t\cdot t}{1-t^2}\,dt=-4\int \frac{t^2}{1-t^2}\,dt=$$

$$=-4\int\left(-1+\frac{1}{1-t^2}\right)dt=4\int dt-4\int \frac{dt}{1-t^2}=4t-4\int \frac{dt}{1-t^2}$$

$$\left\{\begin{array}{l} \frac{1}{1-t^2}=\frac{A}{1-t}+\frac{B}{1+t} \Rightarrow A(1+t)+B(1-t)=1 \Rightarrow A=\frac{1}{2};B=\frac{1}{2} \\ -4\int \frac{dt}{1-t^2}=-2\int \frac{dt}{1-t}-2\int \frac{dt}{1+t}=2Ln|1-t|-2Ln|1+t|+\mathbb{C} \end{array}\right.$$

Deshaciendo los cambios de variable:

$$I=-2\sqrt{1-x}Lnx+4\sqrt{1-x}+2Ln\left|1-\sqrt{1-x}\right|-2Ln\left|1+\sqrt{1-x}\right|+\mathbb{C}$$

$$I=-2\sqrt{1-x}Lnx+4\sqrt{1-x}+2Ln\frac{\left|1-\sqrt{1-x}\right|}{\left|1+\sqrt{1-x}\right|}+\mathbb{C}$$

$$\int \frac{Ln\,x}{\sqrt{1-x}}\,dx=-2\sqrt{1-x}Lnx+4\sqrt{1-x}+2Ln\frac{\left|1-\sqrt{1-x}\right|}{\left|1+\sqrt{1-x}\right|}+\mathbb{C}$$

3 Resuelve la integral:

$$I=\int \frac{x\, arc.\text{tg} x}{(x^2+1)^2}dx$$

SOLUCIÓN

Aplicando partes: $I=\left\{\begin{array}{l} u = arc.\text{tg} x \Rightarrow du = \dfrac{dx}{1+x^2} \\ dv = \dfrac{xdx}{\left(1+x^2\right)^2} \Rightarrow v = -\dfrac{1}{2(1+x^2)} \end{array}\right\} \Rightarrow$

$$I=-\frac{1}{2}\frac{arc.\text{tg} x}{1+x^2}+\frac{1}{2}\int \frac{dx}{(1+x^2)^2}$$

Aplicamos el método de Hermite para resolver: $\int \dfrac{dx}{(1+x^2)^2}$:

$$\int \frac{dx}{(1+x^2)^2}=\frac{ax+b}{1+x^2}+\int \frac{Ax+B}{1+x^2}dx$$

$$\frac{1}{(1+x^2)^2}=\frac{a(1+x^2)-2x(ax+b)}{(1+x^2)^2}+\frac{Ax+B}{1+x^2}\Rightarrow$$

$$1=a(1+x^2)-2x(ax+b)+(Ax+B)(1+x^2)$$

Identificando coeficientes:

$$x^3: \quad 0=A$$

$$x^2: \quad 0=a-2a+B \Rightarrow B=a$$

$$x \;: \quad 0=-2b+A \Rightarrow b=0$$

$$1 \;: \quad 1=a+B \Rightarrow a=B=\frac{1}{2}$$

$$\int \frac{dx}{(1+x^2)^2}=\frac{1}{2}\frac{x}{1+x^2}+\frac{1}{2}\int \frac{dx}{1+x^2}=\frac{1}{2}\frac{x}{1+x^2}+\frac{1}{2}arc.\text{tg} x+C$$

Sustituyendo estos valores resulta que:

$$\int \frac{x\, arc.\text{tg} x}{(x^2+1)^2}dx=-\frac{1}{2}\frac{arc.\text{tg} x}{1+x^2}+\frac{1}{4}\frac{x}{1+x^2}+\frac{1}{4}arc.\text{tg} x+C$$

4 Resuelve la integral:

$$I = \int \frac{\sqrt{x}}{\sqrt[3]{x^2} + 4x}\, dx$$

SOLUCIÓN

Dividimos numerador y denominador por $\sqrt{x}$ y reducimos al mismo índice:

$$I = \int \frac{\sqrt{x}}{\sqrt[3]{x^2} + 4x}\, dx = \int \frac{dx}{x^{1/6} + 4x^{1/2}} = \int \frac{dx}{x^{1/6} + 4x^{3/6}}$$

Hacemos el cambio: $\left\{ \begin{array}{l} x^{1/6} = t \Rightarrow x = t^6 \\ dx = 6t^5 dt \end{array} \right\} \Rightarrow I = \int \frac{6t^5}{t + 4t^3}\, dt =$

$$= 6\int \left(\frac{1}{4}t^2 - \frac{1}{16} + \frac{1}{16}\frac{1}{4t^2 + 1} \right) dt = 6\left(\frac{1}{4 \cdot 3}t^3 - \frac{1}{16}t + \frac{1}{16 \cdot 2} arc.\,\text{tg}(2t) \right) + C$$

deshaciendo el cambio:

$$\int \frac{\sqrt{x}}{\sqrt[3]{x^2} + 4x}\, dx = \frac{1}{2}x^{1/2} - \frac{3}{8}x^{1/6} + \frac{3}{16} arc.\,\text{tg}(2x^{1/6}) + C$$

5 Resuelve la integral:

$$I = \int \frac{thx}{1+thx}dx$$

SOLUCIÓN

Teniendo en cuenta: $\left.\begin{array}{l} \text{sh}x = \dfrac{e^x - e^{-x}}{2} \\ \text{ch}x = \dfrac{e^x + e^{-x}}{2} \end{array}\right\} \Rightarrow \text{th}x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

$$\int \frac{\text{th}x}{1+\text{th}x}dx = \int \frac{\dfrac{e^x - e^{-x}}{e^x + e^{-x}}}{1+\dfrac{e^x - e^{-x}}{e^x + e^{-x}}}dx = \int \frac{e^x - e^{-x}}{2e^x}\,dx = \frac{1}{2}\int dx - \frac{1}{2}\int \frac{e^{-x}}{e^x}dx \Rightarrow$$

$$\int \frac{\text{th}x}{1+\text{th}x}dx = \frac{1}{2}x + \frac{1}{4}e^{-2x} + C$$

6 Resuelve la integral:

$$I = \int \frac{3x^4 + 4x^2 + 2x + 1}{x(x^2+1)^2}\, dx$$

SOLUCIÓN

Puesto que el denominador presenta raíces complejas múltiples, aplicamos el método de Hermite para resolver la integral.

$$\frac{3x^4 + 4x^2 + 2x + 1}{x(x^2+1)^2} = \frac{d}{dx}\left(\frac{ax+b}{x^2+1}\right) + \frac{A}{x} + \frac{Mx+N}{x^2+1}$$

$$\frac{3x^4 + 4x^2 + 2x + 1}{x(x^2+1)^2} = \frac{a(x^2+1) - 2x(ax+b)}{(x^2+1)^2} + \frac{A}{x} + \frac{Mx+N}{x^2+1} \Rightarrow$$

$$3x^4 + 4x^2 + 2x + 1 = ax^3 + ax - 2ax^3 - 2bx^2 + Ax^4 + 2Ax^2 + A + Mx^4 + $$
$$+ Mx^2 + Nx^3 + Nx$$

Igualando los coeficientes de los términos del mismo grado, llegamos al sistema lineal:

$$\begin{aligned} x^4 &: \quad 3 = A + M \\ x^3 &: \quad 0 = -a + N \\ x^2 &: \quad 4 = -2b + 2A + M \\ x &: \quad 2 = a + N \\ 1 &: \quad 1 = A \end{aligned}$$

Su solución es: $a = 1,\ \ b = 0,\ \ A = 1,\ \ M = 2,\ \ N = 1$

En consecuencia: $I = \frac{x}{x^2+1} + \int \frac{dx}{x} + \int \frac{2x}{x^2+1}\, dx + \int \frac{dx}{x^2+1} \Rightarrow$

$$I = \frac{x}{x^2+1} + Ln|x| + Ln|x^2+1| + arc.\text{tg}\, x + C$$

7 Resuelve la integral:

$$I = \int Ln(a^2 + x^2)dx$$

SOLUCIÓN

Aplicamos partes: $\left\{ \begin{array}{ll} u = Ln(a^2 + x^2) \Rightarrow du = \dfrac{2xdx}{a^2 + x^2} \\ dv = dx \qquad \Rightarrow v = x \end{array} \right\} \Rightarrow$

$$I = xLn(a^2 + x^2) - \int \frac{2x^2}{a^2 + x^2} dx$$

$$\int \frac{2x^2}{a^2 + x^2} dx = 2\int \left(1 - \frac{a^2}{a^2 + x^2}\right) dx = 2\int dx - 2\int \frac{dx}{1 + \left(\dfrac{x}{a}\right)^2} \Rightarrow$$

$$\int \frac{2x^2}{a^2 + x^2} dx = 2x - 2a \cdot arc.\text{tg}\left(\frac{x}{a}\right) + C$$

Luego:

$$\int Ln(a^2 + x^2)dx = xLn(a^2 + x^2) - 2x + 2a \cdot arc.\text{tg}\left(\frac{x}{a}\right) + C$$

8 Resuelve la integral:

$$I = \int \operatorname{sen}^2 x \cdot \cos^4 x dx$$

SOLUCIÓN

La integral dada la podemos poner como:

$$I = \int \operatorname{sen}^2 x \cdot \cos^4 x dx = \int \operatorname{sen}^2 x \cdot \cos^2 x . \cos^2 x dx$$

Utilizando las razones trigonométricas:

$$\left.\begin{array}{l} \operatorname{sen} 2x = 2 \operatorname{sen} x \cos x \\ \operatorname{sen}^2 x = \dfrac{1 - \cos 2x}{2} \\ \cos^2 x = \dfrac{1 + \cos 2x}{2} \end{array}\right\}$$ La integral dada se transforma en:

$$I = \frac{1}{4} \int \operatorname{sen}^2 2x \frac{1 + \cos 2x}{2} dx = \frac{1}{8} \int \left(\operatorname{sen}^2 2x + \operatorname{sen}^2 2x \cos 2x \right) dx =$$

$$= \frac{1}{8} \int \left[\frac{1 - \cos 4x}{2} + \frac{1}{2} \left(\operatorname{sen}^2 2x \right) \cos 2x \cdot 2 \right] dx = \frac{1}{16} \left[x - \frac{\operatorname{sen} 4x}{4} + \frac{\operatorname{sen}^3 2x}{3} \right] + C$$

9 Resuelve la integral:

$$I = \int x^2 \operatorname{sen}(Lnx)dx$$

SOLUCIÓN

Aplicamos partes: $\left\{ \begin{array}{ll} u = \operatorname{sen}(Lnx) \Rightarrow du = \dfrac{\cos(Lnx)}{x}dx \\ dv = x^2 dx \qquad \Rightarrow v = \dfrac{x^3}{3} \end{array} \right\} \Rightarrow$

$$I = \frac{x^3 \operatorname{sen}(Lnx)}{3} - \frac{1}{3}\int x^2 \cos(Lnx)dx$$

$$I_1 = \int x^2 \cos(Lnx)dx = \left\{ \begin{array}{ll} u = \cos(Lnx) \Rightarrow du = \dfrac{-\operatorname{sen}(Lnx)}{x}dx \\ dv = x^2 dx \qquad \Rightarrow v = \dfrac{x^3}{3} \end{array} \right\} =$$

$$= \frac{x^3 \cos(Lnx)}{3} + \frac{1}{3}\int x^2 \operatorname{sen}(Lnx)dx$$

$$I = \frac{x^3 \operatorname{sen}(Lnx)}{3} - \frac{1}{9}x^3 \cos(Lnx) - \frac{1}{9}I \Rightarrow$$

$$I = \frac{9}{10}\left[\frac{x^3 \operatorname{sen}(Lnx)}{3} - \frac{1}{9}x^3 \cos(Lnx)\right] + C = \frac{3x^3}{10}\left[\operatorname{sen}(Lnx) - \frac{1}{3}\cos(Lnx)\right] + C$$

10 Resuelve la integral:

$$I = \int \frac{\operatorname{sen} x \cos^2 x}{1 + 4\cos^2 x} dx$$

SOLUCIÓN

Realizamos el cambio de variable: $\left\{ \begin{array}{l} t = \cos x \Rightarrow \operatorname{sen} x = \sqrt{1 - t^2} \\ dx = -\dfrac{dt}{\sqrt{1 - t^2}} \end{array} \right\} \Rightarrow$

$$I = -\int \frac{t^2\sqrt{1 - t^2}}{1 + 4t^2} \frac{dt}{\sqrt{1 - t^2}} = -\frac{1}{4}\int \frac{4t^2}{4t^2 + 1} dt = -\frac{1}{4}\int \frac{4t^2 + 1 - 1}{4t^2 + 1} dt =$$

$$= -\frac{1}{4} t + \frac{1}{4}\int \frac{dt}{(2t)^2 + 1} + \mathbb{C} = -\frac{1}{4} t + \frac{1}{8} arc.\operatorname{tg}(2t) + \mathbb{C} \Rightarrow$$

$$I = -\frac{1}{4}\cos x + \frac{1}{8} arc.\operatorname{tg}(2\cos x) + \mathbb{C}$$

11 Resuelve la integral:

$$I = \int arc.\text{sen}\sqrt{\frac{x}{x+1}}\,dx$$

SOLUCIÓN

Aplicando partes: $\left\{ \begin{array}{l} u = arc.\text{sen}\sqrt{\dfrac{x}{x+1}} \Rightarrow du = \dfrac{dx}{2(x+1)\sqrt{x}} \\ dv = dx \Rightarrow v = x \end{array} \right\} \Rightarrow$

$$I = x\,arc.\text{sen}\sqrt{\frac{x}{x+1}} - \frac{1}{2}\int \frac{xdx}{(x+1)\sqrt{x}}$$

$$I_1 = \int \frac{xdx}{(x+1)\sqrt{x}} = \left\{ \begin{array}{l} x = t^2 \\ dx = 2tdt \end{array} \right\} = \int \frac{t^2}{(t^2+1)t}2tdt = 2\int \frac{t^2}{t^2+1}dt \Rightarrow$$

$$I_1 = 2\int \frac{t^2+1}{t^2+1}dt - 2\int \frac{dt}{t^2+1} = 2t - 2\,arc.\text{tg}\,t + C$$

Por lo tanto:

$$I = x\,arc.\text{sen}\sqrt{\frac{x}{x+1}} - \sqrt{x} + arc.\text{tg}\sqrt{x} + C$$

12 Resuelve la integral:

$$I = \int \frac{2+\mathrm{tg}^2 x}{\left(1+\mathrm{tg}^3 x\right)\cos^2 x}\,dx$$

SOLUCIÓN

Hacemos el cambio: $\left\{\begin{array}{l}\mathrm{tg}\,x = t \\ \dfrac{1}{\cos^2 x}dx = dt\end{array}\right\} \Rightarrow I = \int \dfrac{2+t^2}{\left(1+t^3\right)}dt$

Descomponiendo en fracciones simples:

$$\frac{2+t^2}{\left(1+t^3\right)} = \frac{A}{t+1} + \frac{Mt+N}{t^2-t+1} \Rightarrow 2+t^2 = A\left(t^2-t+1\right) + (Mt+N)(t+1)$$

Igualando los coeficientes de los términos del mismo grado, obtenemos el siguiente sistema lineal:

$$\begin{array}{l} t^2 : \ \ 1 = A + M \\ t \ : \ \ 0 = -A + M + N \\ 1 \ : \ \ 2 = A + N \end{array}$$

Su solución es: $A = 1, \ \ M = 0, \ \ N = 1$

Por lo tanto, tenemos:

$$I = \int \frac{dt}{t+1} + \int \frac{dt}{t^2-t+1} = Ln|t+1| + \int \frac{dt}{t^2-t+1}$$

Teniendo en cuenta que: $ax^2 + bx + c = \left(\sqrt{a}x + \dfrac{b}{2\sqrt{a}}\right)^2 - \dfrac{b^2-4ac}{4a}$ si $a > 0$

$$\int \frac{1}{t^2-t+1}dt = \int \frac{dt}{\left(t-\frac{1}{2}\right)^2 + \frac{3}{4}} = \frac{4}{3}\int \frac{dt}{\left(\frac{2t-1}{\sqrt{3}}\right)^2 + 1} = \frac{2}{\sqrt{3}} arc.\mathrm{tg}\left(\frac{2t-1}{\sqrt{3}}\right) + C$$

$$I = Ln|t+1| + \frac{2}{\sqrt{3}} arc.\mathrm{tg}\left(\frac{2t-1}{\sqrt{3}}\right) + C$$

Deshaciendo el cambio:

$$I = Ln|1+\mathrm{tg}x| + \frac{2}{\sqrt{3}} arc.\mathrm{tg}\left(\frac{2\,\mathrm{tg}x - 1}{\sqrt{3}}\right) + C$$