

Bloque I. Función real de variable real
Tema 6 Integración básica

Ejercicios resueltos

I.6-1 Resuelve las siguientes integrales indefinidas utilizando la propiedad de linealidad y la tabla de integrales inmediatas:

a) $\int (x-2)^2 dx$

b) $\int \frac{1}{x^4} dx$

c) $\int \frac{x^2 - x + 5}{x} dx$

d) $\int (3e^x - \operatorname{sen} x) dx$

e) $\int \sqrt[3]{x^2} dx$

f) $\int \frac{3}{5x^2 + 5} dx$

g) $\int \sqrt{\frac{4}{9-9x^2}} dx$

Solución

$$\begin{aligned} a) \int (x-2)^2 dx &= \int (x^2 - 4x + 4) dx = \int x^2 dx - 4 \int x dx + 4 \int dx = \\ &= \frac{x^3}{3} - 4 \frac{x^2}{2} + 4x + C = \frac{x^3}{3} - 2x^2 + 4x + C \end{aligned}$$

$$b) \int \frac{1}{x^4} dx = \int x^{-4} dx = \frac{x^{-3}}{-3} + C = -\frac{1}{3} \frac{1}{x^3} + C$$

$$\begin{aligned} c) \int \frac{x^2 - x + 5}{x} dx &= \int \left(x - 1 + \frac{5}{x} \right) dx = \int x dx - \int dx + 5 \int \frac{1}{x} dx = \\ &= \frac{x^2}{2} - x + 5 \ln|x| + C \end{aligned}$$

$$d) \int (3e^x - \operatorname{sen} x) dx = 3 \int e^x dx - \int \operatorname{sen} x dx = 3e^x + \cos x + C$$

$$e) \int \sqrt[3]{x^2} dx = \int x^{2/3} dx = \frac{x^{5/3}}{5/3} + C = \frac{3}{5} x^{5/3} + C = \frac{3}{5} \sqrt[3]{x^5} + C = \frac{3}{5} x \sqrt[3]{x^2} + C$$

$$f) \int \frac{3}{5x^2 + 5} dx = \frac{3}{5} \int \frac{1}{x^2 + 1} dx = \frac{3}{5} \operatorname{arctag} x + C$$

$$g) \int \sqrt{\frac{4}{9-9x^2}} dx = \int \sqrt{\frac{4}{9}} \sqrt{\frac{1}{1-x^2}} dx = \sqrt{\frac{4}{9}} \int \frac{1}{\sqrt{1-x^2}} dx = \frac{2}{3} \operatorname{arcsen} x + C$$

I.6-2 Resuelve las siguientes integrales indefinidas utilizando algún cambio de variable apropiado:

a) $\int x\sqrt{x-1}dx$

b) $\int \frac{\operatorname{sen} x}{\sqrt{\cos x}} dx$

c) $\int \frac{x^2}{x^3-2} dx$

d) $\int (e^x - 3)^4 e^x dx$

e) $\int \frac{2x}{1+x^4} dx$

f) $\int \frac{\ln x}{x} dx$

g) $\int \frac{e^{\operatorname{tag} x}}{\cos^2 x} dx$

Solución

$$\begin{aligned} a) \int x\sqrt{x-1}dx &= \left(\begin{array}{l} t = x-1 \\ dt = dx \end{array} \right) = \int (t+1)\sqrt{t}dt = \int t\sqrt{t}dt + \int \sqrt{t}dt = \\ &= \int t \cdot t^{1/2}dt + \int t^{1/2}dt = \int t^{3/2}dt + \int t^{1/2}dt = \frac{t^{5/2}}{5/2} + \frac{t^{3/2}}{3/2} + C = \\ &= \frac{2}{5}t^{5/2} + \frac{2}{3}t^{3/2} + C = \frac{2}{5}(x-1)^{5/2} + \frac{2}{3}(x-1)^{3/2} + C \end{aligned}$$

$$\begin{aligned} b) \int \frac{\operatorname{sen} x}{\sqrt{\cos x}} dx &= \left(\begin{array}{l} t = \cos x \\ dt = -\operatorname{sen} x dx \end{array} \right) = \int \frac{-dt}{\sqrt{t}} = -\int \frac{1}{t^{1/2}} dt = -\int t^{-1/2} dt = \\ &= -\frac{t^{1/2}}{1/2} + C = -2t^{1/2} + C = -2\sqrt{t} + C = -2\sqrt{\cos x} + C \end{aligned}$$

$$c) \int \frac{x^2}{x^3-2} dx = \left(\begin{array}{l} t = x^3-2 \\ dt = 3x^2 dx \end{array} \right) = \frac{1}{3} \int \frac{dt}{t} = \frac{1}{3} \ln|t| + C = \frac{1}{3} \ln|x^3-2| + C$$

$$d) \int (e^x - 3)^4 e^x dx = \left(\begin{array}{l} t = e^x - 3 \\ dt = e^x dx \end{array} \right) = \int t^4 dt = \frac{t^5}{5} + C = \frac{(e^x - 3)^5}{5} + C$$

$$e) \int \frac{2x}{1+x^4} dx = \left(\begin{array}{l} t = x^2 \\ dt = 2x dx \end{array} \right) = \int \frac{1}{1+t^2} dt = \operatorname{arctag} t + C = \operatorname{arctag}(x^2) + C$$

$$f) \int \frac{\ln x}{x} dx = \left(\begin{array}{l} t = \ln x \\ dt = \frac{1}{x} dx \end{array} \right) = \int t dt = \frac{t^2}{2} + C = \frac{(\ln|x|)^2}{2} + C$$

$$g) \int \frac{e^{\operatorname{tag} x}}{\cos^2 x} dx = \left(\begin{array}{l} t = \operatorname{tag} x \\ dt = \frac{1}{\cos^2 x} dx \end{array} \right) = \int e^t dt = e^t + C = e^{\operatorname{tag} x} + C$$

I.6-3 Resuelve las siguientes integrales indefinidas utilizando el método de integración por partes:

a) $\int \ln x dx$

b) $\int x \ln x dx$

c) $\int x e^x dx$

d) $\int x^2 e^x dx$

e) $\int \arcsen x dx$

f) $\int e^x \sen x dx$

g) $\int x^2 \sen x dx$

Solución

$$a) \int \ln x dx = \left(\begin{array}{l} u = \ln x \Rightarrow du = \frac{dx}{x} \\ dv = dx \Rightarrow v = x \end{array} \right) = x \ln x - \int x \frac{dx}{x} = x \ln x - \int dx = \\ = x \ln x - x + C$$

$$b) \int x \ln x dx = \left(\begin{array}{l} u = \ln x \Rightarrow du = \frac{dx}{x} \\ dv = x dx \Rightarrow v = \frac{x^2}{2} \end{array} \right) = \frac{x^2}{2} \ln x - \frac{1}{2} \int x^2 \frac{dx}{x} = \\ = \frac{x^2}{2} \ln x - \frac{1}{2} \int x dx = \frac{x^2}{2} \ln x - \frac{1}{2} \frac{x^2}{2} + C = \frac{x^2}{2} \ln x - \frac{x^2}{4} + C$$

$$c) \int x e^x dx = \left(\begin{array}{l} u = x \Rightarrow du = dx \\ dv = e^x dx \Rightarrow v = e^x \end{array} \right) = x e^x - \int e^x dx = x e^x - e^x + C$$

$$d) \int x^2 e^x dx = \left(\begin{array}{l} u = x^2 \Rightarrow du = 2x dx \\ dv = e^x dx \Rightarrow v = e^x \end{array} \right) = x^2 e^x - 2 \int x e^x dx = \\ = \left(\begin{array}{l} u = x \Rightarrow du = dx \\ dv = e^x dx \Rightarrow v = e^x \end{array} \right) = x^2 e^x - 2 \left[x e^x - \int e^x dx \right] = \\ = x^2 e^x - 2 x e^x + 2 \int e^x dx = x^2 e^x - 2 x e^x + 2 e^x + C$$

$$\begin{aligned}
 e) \int \arcsen x dx &= \left(\begin{array}{l} u = \arcsen x \Rightarrow du = \frac{dx}{\sqrt{1-x^2}} \\ dv = dx \Rightarrow v = x \end{array} \right) = x \arcsen x - \int \frac{x}{\sqrt{1-x^2}} dx = \\
 &= x \arcsen x - \int x (1-x^2)^{-1/2} dx = \left(\begin{array}{l} t = 1-x^2 \\ dt = -2x dx \end{array} \right) = \\
 &= x \arcsen x + \frac{1}{2} \int t^{-1/2} dt = x \arcsen x + \frac{1}{2} \frac{t^{1/2}}{1/2} + C = \\
 &= x \arcsen x + (1-x^2)^{1/2} + C = x \arcsen x + \sqrt{1-x^2} + C
 \end{aligned}$$

$$\begin{aligned}
 f) \int e^x \sen x dx &= \left(\begin{array}{l} u = e^x \Rightarrow du = e^x dx \\ dv = \sen x dx \Rightarrow v = -\cos x \end{array} \right) = -e^x \cos x + \int e^x \cos x dx = \\
 &= \left(\begin{array}{l} u = e^x \Rightarrow du = e^x dx \\ dv = \cos x dx \Rightarrow v = \sen x \end{array} \right) = \\
 &= -e^x \cos x + e^x \sen x - \int e^x \sen x dx \\
 &\Rightarrow 2 \int e^x \sen x dx = -e^x \cos x + e^x \sen x \\
 &\Rightarrow \int e^x \sen x dx = \frac{e^x (\sen x - \cos x)}{2} + C
 \end{aligned}$$

$$\begin{aligned}
 g) \int x^2 \sen x dx &= \left(\begin{array}{l} u = x^2 \Rightarrow du = 2x dx \\ dv = \sen x dx \Rightarrow v = -\cos x \end{array} \right) = -x^2 \cos x + 2 \int x \cos x dx = \\
 &= \left(\begin{array}{l} u = x \Rightarrow du = dx \\ dv = \cos x dx \Rightarrow v = \sen x \end{array} \right) = \\
 &= -x^2 \cos x + 2x \sen x - 2 \int \sen x dx = \\
 &= -x^2 \cos x + 2x \sen x + 2 \cos x + C
 \end{aligned}$$

I.6-4 Resuelve las siguientes integrales indefinidas de funciones racionales:

- a) $\int \frac{x^2+1}{x^3+x^2-2x} dx$
 b) $\int \frac{x^2+1}{x^3-6x^2+12x-8} dx$
 c) $\int \frac{4}{x^3-5x^2+7x-3} dx$
 d) $\int \frac{x^2-3x+2}{x^2-9} dx$
 e) $\int \frac{x^2+x-8}{x^3-4x^2} dx$
 f) $\int \frac{2x^2-x+1}{x^2+8x+16} dx$
 g) $\int \frac{x^4+3x^3-2x^2+1}{x+5} dx$

Solución

a) $\int \frac{x^2+1}{x^3+x^2-2x} dx \Rightarrow x^3+x^2-2x = x(x^2+x-2) = x(x-1)(x+2)$

$$\frac{x^2+1}{x^3+x^2-2x} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+2}$$

$$x^2+1 = A(x-1)(x+2) + Bx(x+2) + Cx(x-1)$$

$$\left. \begin{array}{l} x=0: 1 = -2A \\ x=1: 2 = 3B \\ x=-2: 5 = 6C \end{array} \right\} \Rightarrow \begin{array}{l} A = -1/2 \\ B = 2/3 \\ C = 5/6 \end{array}$$

$$\begin{aligned} \int \frac{x^2+1}{x^3+x^2-2x} dx &= -\frac{1}{2} \int \frac{dx}{x} + \frac{2}{3} \int \frac{dx}{x-1} + \frac{5}{6} \int \frac{dx}{x+2} = \\ &= -\frac{1}{2} \ln|x| + \frac{2}{3} \ln|x-1| + \frac{5}{6} \ln|x+2| + C \end{aligned}$$

b) $\int \frac{x^2+1}{x^3-6x^2+12x-8} dx \Rightarrow x^3-6x^2+12x-8 = (x-2)^3$

$$\frac{x^2+1}{(x-2)^3} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{(x-2)^3}$$

$$x^2+1 = A(x-2)^2 + B(x-2) + C = Ax^2 - 4Ax + 4A + Bx - 2B + C$$

$$\left. \begin{array}{l} x^2: 1 = A \\ x^1: 0 = -4A + B \\ x^0: 1 = 4A - 2B + C \end{array} \right\} \Rightarrow \begin{array}{l} A = 1 \\ B = 4 \\ C = 5 \end{array}$$

$$\begin{aligned} \int \frac{x^2 + 1}{x^3 - 6x^2 + 12x - 8} dx &= \int \frac{dx}{x-2} + 4 \int \frac{dx}{(x-2)^2} + 5 \int \frac{dx}{(x-2)^3} = \\ &= \ln|x-2| - \frac{4}{x-2} - \frac{5}{2(x-2)^2} + C \end{aligned}$$

$$c) \int \frac{4}{x^3 - 5x^2 + 7x - 3} dx \Rightarrow x^3 - 5x^2 + 7x - 3 = (x-1)^2(x-3)$$

$$\frac{4}{(x-1)^2(x-3)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x-3}$$

$$4 = A(x-1)(x-3) + B(x-3) + C(x-1)^2$$

$$\left. \begin{array}{l} x=1: 4 = -2B \\ x=3: 4 = 4C \\ x=0: 4 = 3A - 3B + C \end{array} \right\} \Rightarrow \begin{array}{l} A = -1 \\ B = -2 \\ C = 1 \end{array}$$

$$\begin{aligned} \int \frac{4}{x^3 - 5x^2 + 7x - 3} dx &= - \int \frac{dx}{x-1} - 2 \int \frac{dx}{(x-1)^2} + \int \frac{dx}{x-3} = \\ &= -\ln|x-1| + \frac{2}{x-1} + \ln|x-3| + C \end{aligned}$$

$$d) \int \frac{x^2 - 3x + 2}{x^2 - 9} dx \Rightarrow x^2 - 3x + 2 = 1 \cdot (x^2 - 9) + (-3x + 11)$$

$$\int \frac{x^2 - 3x + 2}{x^2 - 9} dx = \int dx + \int \frac{-3x + 11}{x^2 - 9} dx \Rightarrow x^2 - 9 = (x-3)(x+3)$$

$$\frac{-3x + 11}{x^2 - 9} = \frac{A}{x-3} + \frac{B}{x+3} \Rightarrow -3x + 11 = A(x+3) + B(x-3)$$

$$\left. \begin{array}{l} x=3: 2 = 6A \\ x=-3: 20 = -6B \end{array} \right\} \Rightarrow \begin{array}{l} A = 1/3 \\ B = -10/3 \end{array}$$

$$\begin{aligned} \int \frac{x^2 - 3x + 2}{x^2 - 9} dx &= \int dx + \int \frac{-3x + 11}{x^2 - 9} dx = x + \frac{1}{3} \int \frac{dx}{x-3} - \frac{10}{3} \int \frac{dx}{x+3} = \\ &= x + \frac{1}{3} \ln|x-3| - \frac{10}{3} \ln|x+3| + C \end{aligned}$$

$$e) \int \frac{x^2 + x - 8}{x^3 - 4x^2} dx \Rightarrow x^3 - 4x^2 = x^2(x - 4)$$

$$\frac{x^2 + x - 8}{x^3 - 4x^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x - 4}$$

$$x^2 + x - 8 = Ax(x - 4) + B(x - 4) + Cx^2 = Ax^2 - 4Ax + Bx - 4B + Cx^2$$

$$\left. \begin{array}{l} x^2: 1 = A + C \\ x^1: 1 = -4A + B \\ x^0: -8 = -4B \end{array} \right\} \Rightarrow \begin{array}{l} A = 1/4 \\ B = 2 \\ C = 3/4 \end{array}$$

$$\begin{aligned} \int \frac{x^2 + x - 8}{x^3 - 4x^2} dx &= \frac{1}{4} \int \frac{1}{x} dx + 2 \int \frac{1}{x^2} dx + \frac{3}{4} \int \frac{1}{(x - 4)} dx = \\ &= \frac{1}{4} \ln|x| - \frac{2}{x} + \frac{3}{4} \ln|x - 4| + C \end{aligned}$$

$$f) \int \frac{2x^2 - x + 1}{x^2 + 8x + 16} dx \Rightarrow 2x^2 - x + 1 = 2 \cdot (x^2 + 8x + 16) + (-17x - 31)$$

$$\int \frac{2x^2 - x + 1}{x^2 + 8x + 16} dx = 2 \int dx + \int \frac{-17x - 31}{x^2 + 8x + 16} dx \Rightarrow x^2 + 8x + 16 = (x + 4)^2$$

$$\frac{-17x - 31}{(x + 4)^2} = \frac{A}{x + 4} + \frac{B}{(x + 4)^2} \Rightarrow -17x - 31 = A(x + 4) + B$$

$$\left. \begin{array}{l} x^1: -17 = A \\ x^0: -31 = 4A + B \end{array} \right\} \Rightarrow \begin{array}{l} A = -17 \\ B = 37 \end{array}$$

$$\begin{aligned} \int \frac{2x^2 - x + 1}{x^2 + 8x + 16} dx &= 2 \int dx - 17 \int \frac{1}{x + 4} dx + 37 \int \frac{1}{(x + 4)^2} dx = \\ &= 2x - 17 \ln|x + 4| - \frac{37}{x + 4} + C \end{aligned}$$

$$g) \int \frac{x^4 + 3x^3 - 2x^2 + 1}{x + 5} dx$$

$$x^4 + 3x^3 - 2x^2 + 1 = (x^3 - 2x^2 + 8x - 40) \cdot (x + 5) + 201$$

$$\begin{aligned} \int \frac{x^4 + 3x^3 - 2x^2 + 1}{x + 5} dx &= \int (x^3 - 2x^2 + 8x - 40) dx + \int \frac{201}{x + 5} dx = \\ &= \frac{x^4}{4} - 2 \frac{x^3}{3} + 8 \frac{x^2}{2} - 40x + 201 \ln|x + 5| + C \end{aligned}$$