Ejercicios resueltos

IV.2-1 Resolver las siguientes ecuaciones diferenciales separables:

a) $\frac{dy}{dx}=\frac{x^2-1}{y^2}$ b) $\frac{dy}{dx}=3x^2y$

Solución

a) $\frac{dy}{dx}=\frac{x^2-1}{y^2}\Rightarrow y^2dy=\left(x^2-1\right)dx\Rightarrow\int y^2dy=\int\left(x^2-1\right)dx$

$$\frac{y^3}{3}=\frac{x^3}{3}-x+C\Rightarrow y^3=x^3-3x+C\Rightarrow y=\sqrt[3]{x^3-3x+C}$$

b) $\frac{dy}{dx}=3x^2y\Rightarrow\frac{dy}{y}=3x^2dx\Rightarrow\int\frac{dy}{y}=\int 3x^2dx$

$$\ln\left(y\right)=x^3+C\Rightarrow y=e^{x^3+C}\Rightarrow y=Ke^{x^3}$$

IV.2-2 Resolver el P.V.I. indicado:

a) $\left.\begin{array}{l}2y\frac{dy}{dx}=-x^2\\ y(0)=2\end{array}\right\}$ b) $\left.\begin{array}{l}\frac{dy}{dx}=y\cdot senx\\ y(\pi)=-3\end{array}\right\}$ c) $\left.\begin{array}{l}\frac{dy}{dx}=2\sqrt{y+1}\cdot\cos x\\ y(\pi)=0\end{array}\right\}$

Solución

a) $2y\frac{dy}{dx}=-x^2\Rightarrow 2ydy=-x^2dx\Rightarrow\int 2ydy=-\int x^2dx$

$$y^2=-\frac{x^3}{3}+C\xrightarrow{y(0)=2}4=C\Rightarrow y=\sqrt{-\frac{x^3}{3}+4}$$

b) $\frac{dy}{dx}=y\cdot senx\Rightarrow\frac{dy}{y}=senxdx\Rightarrow\int\frac{dy}{y}=\int senxdx$

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$$\ln(y)=\cos x+C \Rightarrow y=e^{\cos x+C} \Rightarrow y=Ke^{\cos x}$$

$$y=Ke^{\cos x} \xrightarrow{y(\pi)=-3} -3=Ke^{-1} \Rightarrow K=-3e \Rightarrow y=-3e^{1+\cos x}$$

c) $$\frac{dy}{dx}=2\sqrt{y+1}\cdot\cos x \Rightarrow \frac{dy}{\sqrt{y+1}}=2\cos x dx \Rightarrow \int\frac{dy}{\sqrt{y+1}}=\int 2\cos x dx$$

$$\int\frac{dy}{\sqrt{y+1}}=\begin{pmatrix} t=y+1 \\ dt=dy \end{pmatrix}=\int\frac{dt}{\sqrt{t}}=\int t^{-1/2}dt=\frac{t^{1/2}}{1/2}+C=2\sqrt{y+1}+C$$

$$\int 2\cos x dx=2senx+C$$

$$2\sqrt{y+1}=2senx+C \Rightarrow \sqrt{y+1}=senx+C \Rightarrow y=(senx+C)^2-1$$

$$y=(senx+C)^2-1 \xrightarrow{y(\pi)=0} 0=C^2-1 \Rightarrow C=1 \Rightarrow y=(senx+1)^2-1$$

IV.2-3 Resolver las siguientes ecuaciones diferenciales homogéneas:

a) $\dfrac{dy}{dx}=-\dfrac{(x^2+y^2)}{2xy}$ b) $\dfrac{dy}{dx}=\dfrac{xy-y^2}{x^2}$

Solución

a) $$\frac{dy}{dx}=-\frac{(x^2+y^2)}{2xy} \Rightarrow \frac{dy}{dx}=-\frac{\left(1+(y/x)^2\right)}{2(y/x)}$$

$$z=\frac{y}{x} \Rightarrow y=xz \Rightarrow \frac{dy}{dx}=z+x\frac{dz}{dx}$$

$$\frac{dy}{dx}=-\frac{\left(1+(y/x)^2\right)}{2(y/x)} \Rightarrow z+x\frac{dz}{dx}=-\frac{(1+z^2)}{2z} \Rightarrow x\frac{dz}{dx}=-\frac{(1+z^2)}{2z}-z=-\frac{(1+3z^2)}{2z}$$

$$-\frac{2z}{(1+3z^2)}dz=\frac{dx}{x} \Rightarrow -\int\frac{2z}{(1+3z^2)}dz=\int\frac{dx}{x}$$

$$-\int\frac{2z}{(1+3z^2)}dz=\begin{pmatrix} t=1+3z^2 \\ dt=6zdz \end{pmatrix}=-\frac{1}{3}\int\frac{dt}{t}=-\frac{1}{3}\ln(t)+C=-\frac{1}{3}\ln(1+3z^2)+C$$

$$-\frac{1}{3}\ln\left(1+3z^2\right)=\ln(x)+C\Rightarrow\ln\left(1+3z^2\right)=-3\ln(x)+C=-3\ln(Kx)$$

$$1+3z^2=\frac{K}{x^3}\Rightarrow 1+3\left(\frac{y}{x}\right)^2=\frac{K}{x^3}\Rightarrow\frac{x^2+3y^2}{x^2}=\frac{K}{x^3}\Rightarrow x^2+3y^2=\frac{K}{x}$$

$$3y^2=\frac{K}{x}-x^2=\frac{K-x^3}{x}\Rightarrow y^2=\frac{K-x^3}{3x}\Rightarrow y=\sqrt{\frac{K-x^3}{3x}}$$

b) $\frac{dy}{dx}=\frac{xy-y^2}{x^2}\Rightarrow\frac{dy}{dx}=\frac{y}{x}-\left(\frac{y}{x}\right)^2$

$$z=\frac{y}{x}\Rightarrow y=xz\Rightarrow\frac{dy}{dx}=z+x\frac{dz}{dx}$$

$$\frac{dy}{dx}=\frac{y}{x}-\left(\frac{y}{x}\right)^2\Rightarrow z+x\frac{dz}{dx}=z-z^2\Rightarrow x\frac{dz}{dx}=-z^2$$

$$-\frac{dz}{z^2}=\frac{dx}{x}\Rightarrow-\int\frac{dz}{z^2}=\int\frac{dx}{x}\Rightarrow\frac{1}{z}=\ln(x)+C=\ln(Kx)\Rightarrow\frac{x}{y}=\ln(Kx)\Rightarrow y=\frac{x}{\ln(Kx)}$$

IV.2-4 Determinar si las siguientes ecuaciones son exactas. En caso afirmativo, resolverlas:

a) $(2xy+3)dx+\left(x^2-1\right)dy=0$

b) $\frac{1}{y}dx+\left(\frac{x}{y^2}-2y\right)dy=0$

c) $(\cos x\cos y+2x)dx-(senxseny+2y)dy=0$

d) $\cos ydx-\left(xseny-e^y\right)dy=0$

Solución

a) $(2xy+3)dx+\left(x^2-1\right)dy=0$

$$\left.\begin{array}{l}M(x,y)=2xy+3\\N(x,y)=x^2-1\end{array}\right\}\Rightarrow\frac{\partial M(x,y)}{\partial y}=2x=\frac{\partial N(x,y)}{\partial x}\quad\text{EXACTA}$$

$$\frac{\partial F(x,y)}{\partial x} = M(x,y) = 2xy+3 \Rightarrow F(x,y) = \int (2xy+3)\,dx + g(y) = yx^2 + 3x + g(y)$$

$$\frac{\partial F(x,y)}{\partial y} = N(x,y) \Rightarrow x^2 + g'(y) = x^2 - 1 \Rightarrow g'(y) = -1 \Rightarrow g(y) = -y + C$$

$$F(x,y) = yx^2 + 3x - y + C \Rightarrow yx^2 + 3x - y = K \Rightarrow y = \frac{K-3x}{x^2-1}$$

b) $\frac{1}{y}dx + \left(\frac{x}{y^2} - 2y\right)dy = 0$

$$\left.\begin{array}{l} M(x,y) = \dfrac{1}{y} \\ N(x,y) = \dfrac{x}{y^2} - 2y \end{array}\right\} \Rightarrow \left.\begin{array}{l} \dfrac{\partial M(x,y)}{\partial y} = -\dfrac{1}{y^2} \\ \dfrac{\partial N(x,y)}{\partial x} = \dfrac{1}{y^2} \end{array}\right\} \quad \text{NO EXACTA}$$

c) $(\cos x \cos y + 2x)dx - (senxseny + 2y)dy = 0$

$$\left.\begin{array}{l} M(x,y) = \cos x \cos y + 2x \\ N(x,y) = -senxseny - 2y \end{array}\right\} \Rightarrow \left.\begin{array}{l} \dfrac{\partial M(x,y)}{\partial y} = \cos x \\ \dfrac{\partial N(x,y)}{\partial x} = \cos x \end{array}\right\} \quad \text{EXACTA}$$

$$\frac{\partial F(x,y)}{\partial x} = M(x,y) = \cos x \cos y + 2x$$

$$F(x,y) = \int (\cos x \cos y + 2x)\,dx + g(y) = \cos y\, senx + g(y)$$

$$\frac{\partial F(x,y)}{\partial y} = N(x,y)$$

$$-senxseny + g'(y) = -senxseny - 2y \Rightarrow g'(y) = -2y \Rightarrow g(y) = -y^2 + C$$

$$F(x,y) = \cos y\, senx - y^2 + C \Rightarrow \cos y\, senx - y^2 = K$$

d) $\cos y dx - \left(x\operatorname{sen} y - e^{y}\right)dy = 0$

$$\left.\begin{array}{l} M(x,y) = \cos y \\ N(x,y) = -x\operatorname{sen} y + e^{y} \end{array}\right\} \Rightarrow \left.\begin{array}{l} \dfrac{\partial M(x,y)}{\partial y} = -\operatorname{sen} y \\ \dfrac{\partial N(x,y)}{\partial x} = -\operatorname{sen} y \end{array}\right\} \qquad \text{EXACTA}$$

$$\frac{\partial F(x,y)}{\partial x} = M(x,y) = \cos y \Rightarrow F(x,y) = \int \cos y dx + g(y) = x\cos y + g(y)$$

$$\frac{\partial F(x,y)}{\partial y} = N(x,y) \Rightarrow -x\operatorname{sen} y + g'(y) = -x\operatorname{sen} y + e^{y} \Rightarrow g'(y) = e^{y}$$

$$g'(y) = e^{y} \Rightarrow g(y) = e^{y} + C$$

$$F(x,y) = x\cos y + e^{y} + C \Rightarrow x\cos y + e^{y} = K$$

IV.2-5 Resolver el siguiente P.V.I.:

$$\left.\begin{array}{l} \left(e^{x}y+1\right)dx + \left(e^{x}-1\right)dy = 0 \\ y(1) = 1 \end{array}\right\}$$

Solución

$$\left.\begin{array}{l} M(x,y) = e^{x}y+1 \\ N(x,y) = e^{x}-1 \end{array}\right\} \Rightarrow \frac{\partial M(x,y)}{\partial y} = e^{x} = \frac{\partial N(x,y)}{\partial x} \qquad \text{EXACTA}$$

$$\frac{\partial F(x,y)}{\partial x} = M(x,y) \Rightarrow F(x,y) = \int \left(e^{x}y+1\right)dx + g(y) = e^{x}y + x + g(y)$$

$$\frac{\partial F(x,y)}{\partial y} = N(x,y) \Rightarrow e^{x} + g'(y) = e^{x} - 1 \Rightarrow g'(y) = -1 \Rightarrow g(y) = -y + C$$

$$F(x,y) = e^{x}y + x - y + C \Rightarrow e^{x}y + x - y = K \Rightarrow y = \frac{K-x}{e^{x}-1}$$

$$y(1) = 1 \Rightarrow 1 = \frac{K-1}{e-1} \Rightarrow e-1 = K-1 \Rightarrow K = e \Rightarrow y = \frac{e-x}{e^{x}-1}$$

IV.2-6 Considerar la ecuación diferencial $\left(y^2+2xy\right)dx-x^2dy=0$

a) Demostrar que no es exacta.

b) Demostrar que multiplicando ambos miembros de la ecuación por y^{-2} resulta una nueva ecuación que es exacta.

Solución

$$\left.\begin{array}{l} M(x,y)=y^2+2xy \\ N(x,y)=-x^2 \end{array}\right\} \Rightarrow \left.\begin{array}{l} \dfrac{\partial M(x,y)}{\partial y}=2+2x \\ \dfrac{\partial N(x,y)}{\partial x}=0 \end{array}\right\} \quad \text{NO EXACTA}$$

$$\left(1+2\frac{x}{y}\right)dx-\frac{x^2}{y^2}dy=0$$

$$\left.\begin{array}{l} M(x,y)=1+2\dfrac{x}{y} \\ N(x,y)=-\dfrac{x^2}{y^2} \end{array}\right\} \Rightarrow \left.\begin{array}{l} \dfrac{\partial M(x,y)}{\partial y}=-2\dfrac{x}{y^2} \\ \dfrac{\partial N(x,y)}{\partial x}=-2\dfrac{x}{y^2} \end{array}\right\} \quad \text{EXACTA}$$

IV.2-7 Resolver las ecuaciones diferenciales lineales siguientes:

a) $\dfrac{dy}{dx}-y=e^{3x}$

b) $\dfrac{dy}{dx}=x^2e^{-4x}-4y$

Solución

a) $$\frac{dy}{dx}-y=e^{3x} \Rightarrow \left\{\begin{array}{l} P(x)=-1 \\ Q(x)=e^{3x} \end{array}\right\}$$

$$\mu(x)=e^{-\int dx}=e^{-x} \Rightarrow y(x)=e^{x}\left[\int e^{-x}e^{3x}dx+C\right]=e^{x}\left[\int e^{2x}dx+C\right]$$

$$y(x)=e^{x}\left[\frac{e^{2x}}{2}+C\right]=\frac{e^{3x}}{2}+Ce^{x}$$

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b) $\dfrac{dy}{dx}=x^2e^{-4x}-4y\Rightarrow\dfrac{dy}{dx}+4y=x^2e^{-4x}\Rightarrow\begin{Bmatrix}P(x)=4\\Q(x)=x^2e^{-4x}\end{Bmatrix}$

$$\mu(x)=e^{4\int dx}=e^{4x}\Rightarrow y(x)=e^{-4x}\left[\int e^{4x}x^2e^{-4x}dx+C\right]=e^{-4x}\left[\int x^2dx+C\right]$$

$$y(x)=e^{-4x}\left[\frac{x^3}{3}+C\right]$$

IV.2-8 Resolver los siguientes P.V.I.:

a) $\left.\begin{matrix}\dfrac{dy}{dx}-\dfrac{y}{x}=xe^x\\ y(1)=e-1\end{matrix}\right\}$ b) $\left.\begin{matrix}senx\dfrac{dy}{dx}+y\cos x=xsenx\\ y(\pi/2)=2\end{matrix}\right\}$

Solución

a) $\dfrac{dy}{dx}-\dfrac{y}{x}=xe^x\Rightarrow\begin{Bmatrix}P(x)=-\dfrac{1}{x}\\Q(x)=xe^x\end{Bmatrix}$

$$\mu(x)=e^{-\int\frac{dx}{x}}=e^{-\ln(x)}=\frac{1}{x}\Rightarrow y(x)=x\left[\int\frac{1}{x}xe^xdx+C\right]=x\left[\int e^xdx+C\right]$$

$$y(x)=x\left[e^x+C\right]\xrightarrow{y(1)=e-1}e-1=e+C\Rightarrow C=-1\Rightarrow y(x)=x\left[e^x-1\right]$$

b) $senx\dfrac{dy}{dx}+y\cos x=xsenx\Rightarrow\dfrac{dy}{dx}+\dfrac{\cos x}{senx}y=x\begin{Bmatrix}P(x)=\dfrac{\cos x}{senx}\\Q(x)=x\end{Bmatrix}$

$$\mu(x)=e^{\int\frac{\cos x}{senx}dx}=e^{\ln(senx)}=senx\Rightarrow y(x)=\frac{1}{senx}\left[\int xsenxdx+C\right]$$

$$\int xsenxdx=\begin{pmatrix}u=x & \Rightarrow & du=dx\\ dv=senxdx & \Rightarrow & v=-\cos x\end{pmatrix}=-x\cos x+\int\cos xdx=-x\cos x+senx$$

$$y(x)=\frac{1}{senx}\left[-x\cos x+senx+C\right]\xrightarrow{y(\pi/2)=2}2=1+C\Rightarrow C=1$$

$$y(x)=\frac{1}{senx}\left[-x\cos x+senx+1\right]$$