Tema 4 Métodos de Aproximación Numérica

Ejercicios resueltos

IV.4-1 Usar el método de Euler para aproximar la solución del P.V.I. dado en los puntos x = 0.1, 0.2, 0.3, 0.4, 0.5 usando tamaño de paso h = 0.1.

a) $\left.\begin{array}{l} \dfrac{dy}{dx} = -\dfrac{x}{y} \\ y(0) = 4 \end{array}\right\}$ b) $\left.\begin{array}{l} \dfrac{dy}{dx} = x + y \\ y(0) = 1 \end{array}\right\}$

Solución

$$y_{n+1} = y_n + h \cdot f(x_n, y_n)$$

a) $\left.\begin{array}{l} \dfrac{dy}{dx} = -\dfrac{x}{y} \\ y(0) = 4 \end{array}\right\}$

$x_0 = 0 \qquad y_0 = 4$

$x_1 = 0{,}1 \qquad y_1 = y_0 + h \cdot f(x_0, y_0) = 4 + 0{,}1 \cdot \left(-\dfrac{0}{4}\right) = 4$

$x_2 = 0{,}2 \qquad y_2 = y_1 + h \cdot f(x_1, y_1) = 4 + 0{,}1 \cdot \left(-\dfrac{0{,}1}{4}\right) = 3{,}9975$

$x_3 = 0{,}3 \qquad y_3 = y_2 + h \cdot f(x_2, y_2) = 3{,}9975 + 0{,}1 \cdot \left(-\dfrac{0{,}2}{3{,}9975}\right) = 3{,}9925$

$x_4 = 0{,}4 \qquad y_4 = y_3 + h \cdot f(x_3, y_3) = 3{,}9925 + 0{,}1 \cdot \left(-\dfrac{0{,}3}{3{,}9925}\right) = 3{,}2411$

$x_5 = 0{,}5 \qquad y_5 = y_4 + h \cdot f(x_4, y_4) = 3{,}2411 + 0{,}1 \cdot \left(-\dfrac{0{,}4}{3{,}2411}\right) = 3{,}2288$

b) $\left.\begin{array}{l} \dfrac{dy}{dx} = x + y \\ y(0) = 1 \end{array}\right\}$

$x_0 = 0 \qquad y_0 = 1$

$x_1 = 0{,}1 \qquad y_1 = y_0 + h \cdot f(x_0, y_0) = 1 + 0{,}1 \cdot (0+1) = 1{,}1$

$x_2 = 0{,}2 \qquad y_2 = y_1 + h \cdot f(x_1, y_1) = 1{,}1 + 0{,}1 \cdot (0{,}1 + 1{,}1) = 1{,}22$

$x_3 = 0{,}3 \qquad y_3 = y_2 + h \cdot f(x_2, y_2) = 1{,}22 + 0{,}1 \cdot (0{,}2 + 1{,}22) = 1{,}362$

$x_4 = 0{,}4 \qquad y_4 = y_3 + h \cdot f(x_3, y_3) = 1{,}362 + 0{,}1 \cdot (0{,}3 + 1{,}362) = 1{,}5282$

$x_5 = 0{,}5 \qquad y_5 = y_4 + h \cdot f(x_4, y_4) = 1{,}5282 + 0{,}1 \cdot (0{,}4 + 1{,}5282) = 1{,}72102$

IV.4-2 Usar el método de Euler para aproximar la solución del P.V.I. dado en x = 1. Tomar diferentes pasos, h = 1, 0.5, 0.25.

$$\left.\begin{array}{l} \dfrac{dy}{dx} = 1 + x sen(xy) \\ y(0) = 0 \end{array}\right\}$$

Solución

h = 1

$x_0 = 0 \qquad y_0 = 0$

$x_1 = 1 \qquad y_1 = y_0 + h \cdot f(x_0, y_0) = 0 + 1 \cdot (1+0) = 1$

h = 0.5

$x_0 = 0 \qquad y_0 = 0$

$x_1 = 0{,}5 \qquad y_1 = y_0 + h \cdot f(x_0, y_0) = 0 + 0{,}5 \cdot (1+0) = 0{,}5$

$x_2 = 1$

$y_2 = y_1 + h \cdot f(x_1, y_1) = 0{,}5 + 0{,}5 \cdot \left(1 + 0{,}5 \cdot sen(0{,}5 \cdot 0{,}5)\right) = 1{,}06185$

h = 0.25

$x_0 = 0 \qquad y_0 = 0$

$x_1 = 0{,}25 \qquad y_1 = y_0 + h \cdot f(x_0, y_0) = 0 + 0{,}25 \cdot (1+0) = 0{,}25$

$$x_2 = 0,5$$
$$y_2 = y_1 + h \cdot f(x_1, y_1) = 0,25 + 0,25 \cdot (1 + 0,25 \cdot sen(0,25 \cdot 0,25)) = 0{,}503904$$

$$x_3 = 0,75$$
$$y_3 = y_2 + h \cdot f(x_2, y_2) = 0,503904 + 0,25 \cdot (1 + 0,5 \cdot sen(0,5 \cdot 0,503904)) = 0{,}785066$$

$$x_4 = 1$$
$$y_4 = y_3 + h \cdot f(x_3, y_3) = 0,785066 + 0,25 \cdot (1 + 0,75 \cdot sen(0,75 \cdot 0,785066)) = 1{,}1392$$

IV.4-3 Usar el método de Euler mejorado con tamaño de paso h = 0.1 para aproximar la solución del P.V.I. dado en los puntos x = 1.1, 1.2, 1.3, 1.4, 1.5.

$$\left.\begin{array}{l} \dfrac{dy}{dx} = x - y^2 \\ y(1) = 0 \end{array}\right\}$$

Solución

$$y_{n+1} = y_n + \frac{h}{2} \cdot \left[f(x_n, y_n) + f\left(x_n + h, y_n + hf(x_n, y_n)\right) \right]$$

$$x_0 = 1 \qquad y_0 = 0$$

$$x_1 = 1{,}1 \qquad y_1 = 0 + 0,05 \cdot \left[1 + 1{,}1 - 0.1^2\right] = 0.1045$$

$$x_2 = 1,2 \qquad y_2 = 0,1045 + 0,05 \cdot \left[1{,}1 - (0{,}1045)^2 + f(1{,}2; 0{,}213408)\right]$$
$$y_2 = 0,1045 + 0,05 \cdot \left[1{,}1 - (0{,}1045)^2 + 1{,}2 - (0{,}213408)^2\right] = 0{,}216677$$

$$x_3 = 1,3 \qquad y_3 = 0,216677 + 0,05 \cdot \left[1{,}2 - (0{,}216677)^2 + f(1{,}3; 0{,}331982)\right]$$
$$y_3 = 0,216677 + 0,05 \cdot \left[1{,}2 - (0{,}216677)^2 + 1{,}3 - (0{,}331982)^2\right] = 0{,}333819$$

$$x_4 = 1,4 \qquad y_4 = 0,333819 + 0,05 \cdot \left[1{,}3 - (0{,}333819)^2 + f(1{,}4; 0{,}452675)\right]$$
$$y_4 = 0,333819 + 0,05 \cdot \left[1{,}3 - (0{,}333819)^2 + 1{,}4 - (0{,}452675)^2\right] = 0{,}453002$$

$$x_5 = 1,5 \qquad y_5 = 0,453002 + 0,05 \cdot \left[1{,}4 - (0{,}453002)^2 + f(1{,}5; 0,46495)\right]$$
$$y_5 = 0,453002 + 0,05 \cdot \left[1{,}4 - (0{,}453002)^2 + 1{,}5 - (0{,}46495)^2\right] = 0{,}465395$$

IV.4-4 Usar el algoritmo de Euler mejorado para aproximar la solución del P.V.I. dado en x = 1 con tamaño de paso 0.25.

$$\left.\begin{array}{l}\dfrac{dy}{dx} = 1 - y - y^3 \\ y(0) = 0\end{array}\right\}$$

Solución

$$y_{n+1} = y_n + \frac{h}{2}\cdot\left[f(x_n, y_n) + f\left(x_n + h, y_n + hf(x_n, y_n)\right)\right]$$

$$x_0 = 0 \qquad\qquad y_0 = 0$$

$$x_1 = 0{,}25$$

$$y_1 = 0 + \frac{0{,}25}{2}\cdot\left[1 + f(0{,}25; 0{,}25)\right] = 0{,}125\cdot\left[1 + 1 - 0{,}25 - (0{,}25)^3\right] = 0{,}216797$$

$$x_2 = 0{,}5 \qquad y_2 = 0{,}216797 + \frac{0{,}25}{2}\cdot\left[1 - 0{,}216797 - 0{,}216797^3 + f(0{,}5; 0{,}41005)\right]$$

$$y_2 = 0{,}216797 + \frac{0{,}25}{2}\cdot\left[1 - 0{,}216797 - 0{,}216797^3 + 1 - 0{,}41005 - 0{,}41005^3\right] = 0.378549$$

$$x_3 = 0{,}75 \qquad y_3 = 0{,}378549 + \frac{0{,}25}{2}\cdot\left[1 - 0{,}378549 - 0{,}378549^3 + f(0{,}75; 0{,}52035)\right]$$

$$y_3 = 0{,}378549 + \frac{0{,}25}{2}\cdot\left[1 - 0{,}378549 - 0{,}378549^3 + 1 - 0{,}52035 - 0{,}52035^3\right] = 0{,}491794$$

$$x_4 = 1 \qquad y_4 = 0{,}491794 + \frac{0{,}25}{2}\cdot\left[1 - 0{,}491794 - 0{,}491794^3 + f(1; 0{,}589109)\right]$$

$$y_4 = 0{,}491794 + \frac{0{,}25}{2}\cdot\left[1 - 0{,}491794 - 0{,}491794^3 + 1 - 0{,}589109 - 0{,}589109^3\right] = 0{,}566257$$

IV.4-5 Determinar las fórmulas recursivas del método de Taylor de orden 2 para el P.V.I.

$$\left.\begin{array}{l}\dfrac{dy}{dx} = \cos(x + y) \\ y(0) = \pi\end{array}\right\}$$

Solución

$$y_{n+1} = y_n + h\cdot f(x_n, y_n) + \frac{h^2}{2!}\cdot f_2(x_n, y_n) + \ldots + \frac{h^p}{p!}\cdot f_p(x_n, y_n)$$

$$f_2(x_n,y_n) = y''(x) = \left(\cos(x+y)\right)' = -\left(1+y'\right) sen(x+y) =$$
$$= -\left(1+\cos(x+y)\right) sen(x+y) = -sen(x+y) - \cos(x+y) sen(x+y)$$

$$y_{n+1} = y_n + h\cdot\cos(x_n+y_n) - \frac{h^2}{2!}\left(1+\cos(x_n+y_n)\right) sen(x_n+y_n)$$

IV.4-6 Usar el método de Taylor de orden 2 con h = 0.25 para aproximar la solución del P.V.I. dado en x = 1.

$$\left.\begin{array}{l}\dfrac{dy}{dx} = x+1-y \\ y(0)=1\end{array}\right\}$$

Comparar esta aproximación con la solución verdadera, $y = x + e^{-x}$, evaluada en x = 1.

Solución

$$y_{n+1} = y_n + h\cdot f(x_n,y_n) + \frac{h^2}{2!}\cdot f_2(x_n,y_n)$$

$$f_2(x_n,y_n) = y''(x) = (x+1-y)' = \left(1-y'\right) = -x+y$$

$x_0 = 0$ $\qquad$ $y_0 = 1$

$x_1 = 0{,}25$ $\qquad$ $y_1 = y_0 + h\cdot f(x_0,y_0) + \frac{h^2}{2!}\cdot f_2(x_0,y_0) = 1{,}03125$

$x_2 = 0{,}5$ $\qquad$ $y_2 = y_1 + h\cdot f(x_1,y_1) + \frac{h^2}{2!}\cdot f_2(x_1,y_1) = 1{,}11035$

$x_3 = 0{,}75$ $\qquad$ $y_3 = y_2 + h\cdot f(x_2,y_2) + \frac{h^2}{2!}\cdot f_2(x_2,y_2) = 1{,}22684$

$x_4 = 1$ $\qquad$ $y_4 = y_3 + h\cdot f(x_3,y_3) + \frac{h^2}{2!}\cdot f_2(x_3,y_3) = 1{,}37253$

$$y = x + e^{-x} \Rightarrow y(1) = 1 + e^{-1} = 1{,}36788$$

IV.4-7 Usar el método de Runge-Kutta de cuarto orden con h = 0.25 para aproximar la solución del P.V.I. dado en x = 1:

$$\left.\begin{array}{l}\dfrac{dy}{dx}=2y-6\\ y(0)=1\end{array}\right\}$$

Comparar esta aproximación con la solución verdadera, $y=3-2e^{2x}$, evaluada en x = 1.

Solución

$$\left.\begin{array}{l}x_{n+1}=x_n+h\\ y_{n+1}=y_n+\dfrac{1}{6}\cdot(k_1+2k_2+2k_3+k_4)\end{array}\right\}\qquad \left.\begin{array}{l}k_1=h\cdot f(x_n,y_n)\\ k_2=h\cdot f\left(x_n+\dfrac{h}{2},y_n+\dfrac{k_1}{2}\right)\\ k_3=h\cdot f\left(x_n+\dfrac{h}{2},y_n+\dfrac{k_2}{2}\right)\\ k_4=h\cdot f(x_n+h,y_n+k_3)\end{array}\right\}$$

n = 0

$x_0=0 \qquad y_0=1$

n = 1

$x_1=0,25 \qquad y_1=y_0+\dfrac{1}{6}\cdot(k_1+2k_2+2k_3+k_4)=-0,296875$

$$\left.\begin{array}{l}k_1=h\cdot f(x_0,y_0)=-1\\ k_2=h\cdot f\left(x_0+\dfrac{h}{2},y_0+\dfrac{k_1}{2}\right)=-1,25\\ k_3=h\cdot f\left(x_0+\dfrac{h}{2},y_0+\dfrac{k_2}{2}\right)=-1,3125\\ k_4=h\cdot f(x_0+h,y_0+k_3)=-1,65625\end{array}\right\}$$

n = 2

$x_2=0,5 \qquad y_2=y_1+\dfrac{1}{6}\cdot(k_1+2k_2+2k_3+k_4)=-2,434692$

$$\left.\begin{array}{l}k_1=h\cdot f(x_1,y_1)=-1,6484375\\ k_2=h\cdot f\left(x_1+\dfrac{h}{2},y_1+\dfrac{k_1}{2}\right)=-2,06055\\ k_3=h\cdot f\left(x_1+\dfrac{h}{2},y_1+\dfrac{k_2}{2}\right)=-2,1636\\ k_4=h\cdot f(x_1+h,y_1+k_3)=-2,7302\end{array}\right\}$$

n = 3

$x_3=0,75 \qquad y_3=y_2+\dfrac{1}{6}\cdot(k_1+2k_2+2k_3+k_4)=-5,95875$

$$\left.\begin{aligned}
k_1 &= h \cdot f(x_2, y_2) = -2{,}71735 \\
k_2 &= h \cdot f\left(x_2 + \frac{h}{2}, y_2 + \frac{k_1}{2}\right) = -3{,}39668 \\
k_3 &= h \cdot f\left(x_2 + \frac{h}{2}, y_2 + \frac{k_2}{2}\right) = -3{,}5665 \\
k_4 &= h \cdot f(x_2 + h, y_2 + k_3) = -4{,}5006
\end{aligned}\right\}$$

n = 4

$x_4 = 1$ $\qquad y_4 = y_3 + \frac{1}{6} \cdot (k_1 + 2k_2 + 2k_3 + k_4) = -11{,}7679$

$$\left.\begin{aligned}
k_1 &= h \cdot f(x_3, y_3) = -4{,}47938 \\
k_2 &= h \cdot f\left(x_3 + \frac{h}{2}, y_3 + \frac{k_1}{2}\right) = -5{,}5992 \\
k_3 &= h \cdot f\left(x_3 + \frac{h}{2}, y_3 + \frac{k_2}{2}\right) = -5{,}8792 \\
k_4 &= h \cdot f(x_3 + h, y_3 + k_3) = -7{,}4189
\end{aligned}\right\}$$

$y = 3 - 2e^{2x} \Rightarrow y(1) = 3 - 2e^2 = -11{,}7781$

IV.4-8 Usar el método de Runge-Kutta de cuarto orden con h = 0.25 para aproximar la solución del P.V.I. dado en x = 1.

$$\left.\begin{aligned}
\frac{dy}{dx} &= x + 1 - y \\
y(0) &= 1
\end{aligned}\right\}$$

Solución

$$\left.\begin{aligned}
x_{n+1} &= x_n + h \\
y_{n+1} &= y_n + \frac{1}{6} \cdot (k_1 + 2k_2 + 2k_3 + k_4)
\end{aligned}\right\}
\qquad
\left.\begin{aligned}
k_1 &= h \cdot f(x_n, y_n) \\
k_2 &= h \cdot f\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right) \\
k_3 &= h \cdot f\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right) \\
k_4 &= h \cdot f(x_n + h, y_n + k_3)
\end{aligned}\right\}$$

n = 0

$x_0 = 0$ $\qquad y_0 = 1$

n = 1

$x_1 = 0{,}25$ $\qquad y_1 = y_0 + \frac{1}{6} \cdot (k_1 + 2k_2 + 2k_3 + k_4) = 1{,}0288$

$$\left.\begin{aligned}
k_1 &= h\cdot f(x_0,y_0)=0\\
k_2 &= h\cdot f\left(x_0+\frac{h}{2},y_0+\frac{k_1}{2}\right)=0,03125\\
k_3 &= h\cdot f\left(x_0+\frac{h}{2},y_0+\frac{k_2}{2}\right)=0,02734\\
k_4 &= h\cdot f(x_0+h,y_0+k_3)=0,05566
\end{aligned}\right\}$$

n = 2

$x_2=0,5$ $$y_2=y_1+\frac{1}{6}\cdot(k_1+2k_2+2k_3+k_4)=1,10654$$

$$\left.\begin{aligned}
k_1 &= h\cdot f(x_1,y_1)=0,05529\\
k_2 &= h\cdot f\left(x_1+\frac{h}{2},y_1+\frac{k_1}{2}\right)=0,07963\\
k_3 &= h\cdot f\left(x_1+\frac{h}{2},y_1+\frac{k_2}{2}\right)=0,07659\\
k_4 &= h\cdot f(x_1+h,y_1+k_3)=0,09864
\end{aligned}\right\}$$

n = 3

$x_3=0,75$ $$y_3=y_2+\frac{1}{6}\cdot(k_1+2k_2+2k_3+k_4)=1,22238$$

$$\left.\begin{aligned}
k_1 &= h\cdot f(x_2,y_2)=0,098364\\
k_2 &= h\cdot f\left(x_2+\frac{h}{2},y_2+\frac{k_1}{2}\right)=0,117318\\
k_3 &= h\cdot f\left(x_2+\frac{h}{2},y_2+\frac{k_2}{2}\right)=0,114949\\
k_4 &= h\cdot f(x_2+h,y_2+k_3)=0,122126
\end{aligned}\right\}$$

n = 4

$x_4=1$ $$y_4=y_3+\frac{1}{6}\cdot(k_1+2k_2+2k_3+k_4)=1,36789$$

$$\left.\begin{aligned}
k_1 &= h\cdot f(x_3,y_3)=0,1319\\
k_2 &= h\cdot f\left(x_3+\frac{h}{2},y_3+\frac{k_1}{2}\right)=0,14666\\
k_3 &= h\cdot f\left(x_3+\frac{h}{2},y_3+\frac{k_2}{2}\right)=0,14482\\
k_4 &= h\cdot f(x_3+h,y_3+k_3)=0,15819
\end{aligned}\right\}$$