UNIT 8. NON-LINEAR REGRESSION MODELS

NON-LINEAR REGRESSION

Sometimes the scatter plot suggests that the curve Y=(X) is non-linear. In that case the relationship between the variables may be represented by another type of mathematical function, e.g. polynomial, power, exponential...

In order to get the parameters of such a function, the least squares method requires adapting the expression of the Mean Square Error to the appropriate equation and then performing the corresponding optimization.

$$\boldsymbol{MSE} = \sum_{i=1}^{N} (y_i - \hat{y}_i)^2 = \sum_{i=1}^{N} (y_i - f(x_i)\,)^2$$

Nevertheless, there are non-linear functions with the particularity that their equations may be “reformulated” in the form of a linear equation, although using other variables. This process is known as linearisation and involves a change of variable. To estimate the regression function for this type of functions, it is possible to use a linear regression between the transformed variables. The expressions of the parameters of the linear model may be used to determine the non-linear regression function.

For example, if the function to be fitted is an exponential, i.e. $Y = ae^{bX}$, taking logarithms in both members we obtain the following:

$$\ln Y = \ln a + bX$$

so that, naming $Y' = \ln Y$ and $a' = \ln a$, it is obtained:

$$Y' = a' + bX$$

which is the expression of a simple linear regression model.

Analogously, if the function to be fitted is a potential of the type $Y = aX^b$, again taking logarithms in both members, it is obtained:

$$\ln Y = \ln a + b \ln X$$

so that, naming $Y' = \ln Y$, $a' = \ln a$ and $X' = \ln X$, we obtain:

$$Y' = a' + bX'$$

which is again the expression of a simple linear regression model.

In the following, some non-linear regression models and the transformations that would have to be made to apply linear regression techniques to estimate those models are given:

- **Hyperbolic** regression model:

$$Y = a + \frac{b}{X} \Leftrightarrow Y = a + bX'$$
$$\text{with} \quad X' = \frac{1}{X}$$

- **Logarithmic** regression model:

$$Y = a + b \ln X \Leftrightarrow Y = a + bX'$$
$$\text{with} \quad X' = \ln X$$

- **Exponential** regression model:

$$Y = ae^{bX} \Leftrightarrow \ln Y = \ln a + bX \Leftrightarrow Y' = a' + bX$$
$$\text{with} \quad Y' = \ln Y$$
$$a' = \ln a$$

- **Power** regression model:

$$\boldsymbol{Y = aX^b \Leftrightarrow \ln Y = \ln a + b \ln X \Leftrightarrow Y' = a' + bX'}$$
$$\textit{with} \quad \boldsymbol{X' = \ln X \,; \quad Y' = \ln Y}$$
$$\boldsymbol{a' = \ln a}$$

INTERPRETATION OF REGRESSION RESULTS

The results obtained in a regression are of little value if they are not interpreted in the context in which the model has been proposed. This interpretation must take into account, on the one hand, the equation of the model and, on the other, the variables used, *X* and *Y*, in the study. Typically, it is a model that seeks to analyse the impact exerted by the independent variable *X* on the dependent variable *Y* as well as its explanatory power.

The impact is usually studied by analysing the value and sign of the regression coefficient *b* as well as the shape of the estimated line. The coefficient a is an estimation of *Y* for a specific value of *X* (usually 0 or 1) and its economic meaning depends on whether the predicted value corresponds to an interpolation or an extrapolation. In the latter case, its validity is usually more questionable the higher the degree of extrapolation associated with the prediction made. The explanatory power of the model is analysed, in relative terms, with the coefficient of determination and, in absolute terms, using the standard deviation of the residual.

In the following table, we give some guidelines on how to interpret some of the coefficients of the above models.

Table: Interpretation of regression coefficients

Name	Equation	a	b
Linear	$Y = a + bX$	Predicted value of Y for $X = 0$	**Expected growth** of $\boldsymbol{Y}$ when $\boldsymbol{X}$ **increases by 1 unit**.
Exponential	$Y = ab^X$	Predicted value of Y for $X = 0$	**Expected grow** of $\boldsymbol{Y}$ in **relative terms** when $\boldsymbol{X}$ **increases by 1 unit**. The model is usually associated with phenomena in which increasing marginal returns are involved.
Logarithmic	$Y = a + b \ln X$	Predicted value of Y for $X = 1$	**Expected growth** of $\boldsymbol{Y}$ when $\boldsymbol{X}$ **increases by 1 percent unit**. The model is usually associated with phenomena in which decreasing marginal returns are involved.
Power	$Y = aX^b$	Predicted value of Y for $X = 1$	**Elasticity** of Y regarding X.
Hyperbolic	$Y = a + \frac{b}{X}$	Asymptotic value of Y when $X \to \infty$	The model is usually associated with phenomena in which decreasing marginal returns with asymptotic values are involved and determines the pace at which the limit is reached.

With regard to the explanatory power of the model, models with coefficients of determination higher than 75% are usually considered good in relative terms, and those with coefficients higher than 50% are considered average fits. However, the coefficient of determination alone is not enough. The standard deviation of the residuals determines the validity of the regression model in the context of the problem. This validity requires a specific knowledge of the problem, as well as the goals of the research.